Algebraic Semantics of Refinement Modal Logic

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Abstract

We develop algebraic semantics of refinement modal logic using duality theory. Refinement modal logic has quantifiers that are interpreted using a refinement relation. A refinement relation is like a bisimulation, except that from the three relational requirements only 'atoms' and 'back' have to be satisfied. We study the dual notion of refinement on algebras and present algebraic semantics of refinement modal logic. To this end, we first present the algebraic semantics of action model logic quantifier, and we then introduce an algebraic model based on the semantics of the refinement quantifier in terms of the refinement relation. Then we show that refinement modal logic is sound and complete with respect to this algebraic semantics.

Keywords: Refinement modal logic, arbitrary action model logic, dynamic epistemic logic, algebraic semantics.

1 Introduction

In *modal logic* we attempt to formalize propositions about *possibility* and *necessity*. In epistemic modal logics the modal operator is interpreted as knowledge or belief [19], initially for a single knowing agent but later for a set of agents, including their higher-order knowledge (i.e., what they know about each other) [11]. The knowledge of agents is encoded in a relational structure known as a *Kripke model* or *relational structure*, consisting of a domain of worlds, a binary accessibility relation for each agent, and a valuation of atomic propositions over the worlds. Informative updates can be formalized as yet another modal operator, a dynamic modality, that is interpreted as a relation between such Kripke models. A well-known form of informative updates are action models [5], wherein the updates themselves also take the shape of a relational structure.

The Kripke model resulting from executing an action model in an initial Kripke model can also can be seen as a *refinement* of that initial model. A

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refinement relation is like a bisimulation relation, except that from the three relational requirements only 'atoms' and 'back' need to be satisfied. This therefore results in structural loss. From the perspective of knowledge change, this implies that in the refined model agents know more, namely they are less uncertain between different worlds. In [7] refinement modal logic (RML) is introduced, wherein modal logic is augmented with a new operator $\exists$ (and with its dual $\forall$, they are interdefinable as usual), which quantifies over all refinements of a given pointed model. In this logic the expression $\exists\varphi$ stands for "there is a refinement after which φ." In other words, $\exists\varphi$ is true in a Kripke model M with point s (we write M_s for such a pair) if there is a pointed model $M'_{s'}$ such that $(M_s, M'_{s'})$ are a pair in the refinement relation, (we also say that $M'_{s'}$ is a refinement of M_s), and such that φ is true in $M'_{s'}$. The logic is equally expressive as basic modal logic. A well-known result is that action model execution results in a refinement. We can similarly (although not trivially) augment the logic of knowledge with refinement quantifiers, and also the multi-agent logic of knowledge.

A different form of quantification is over action models. This has been investigated in [18]. This logic is called arbitrary action model logic. It is an extension of action model logic with an action model quantifier such that $\exists\varphi$ stands for "there is an action model such that after its execution φ (is true)." Given such an expression $\exists\varphi$, in [18] Hales presents a method for synthesizing a multi-pointed action model α_{T} after which φ is true (in the sense that $\exists\varphi$ is logically equivalent to $\langle\alpha_{\mathsf{T}}\rangle\varphi$), and he also proved that the action model quantifier is equivalent to the refinement quantifier.

In this paper we develop an algebraic semantics of refinement modal logic. Already from close to the inception of dynamic epistemic logics, there has been a strong current to model such logics in algebraic or coalgebraic settings [3,4]. More recently, in [21,22] an algebraic semantics was proposed for public announcement logic and action model logic. This methodology has further been productively used in [9] for a probabilistic dynamic epistemic logic and in [2] for epistemic updates on bilattices.

In [21,22], product updates are dually characterized through a construction that transforms the complex algebra associated with a given Kripke model into the complex algebra associated with the model updated by means of an action model. Given a Kripke model M and an action model α, the result of executing that action model can be seen as a submodel of a so-called intermediate model that contains copies of M indexed by the domain of α. In this way, action model logic can be endowed with an algebraic semantics that is dual (and equivalent) to the relational one, via a Jónsson-Tarski-type duality [6]. In particular, this holds for the multi-pointed action model α_{T} such that $\exists\varphi$ is equivalent to $\langle\alpha_{\mathsf{T}}\rangle\varphi$, according [18] mentioned above.

We use this result to define the algebraic semantics of RML. Indeed, we can dually characterize the algebraic notion of refinement relation as a lax-morphism (named *refinement morphism*) between the complex algebras associated with a given initial Kripke model and a 'resulting' Kripke model that is in the refinement relation with the initial model. Then, via the Jónsson-Tarski duality, we associate that resulting Kripke model to a boolean algebra with operators (BAO). Given the set of all refinements of the initial Kripke model, we then take the product of all corresponding BAOs in order to define a unique algebra and the required refinement morphism. The motivation behind our approach is to capture the non-constructive notion of refinement. Whereas arbitrary action model logic approaches the notion of refinement with brute force by having a witnessing action model that enforces the same post-condition φ bound by the quantifier, refinement modal logic only needs the existence of such an epistemic action (and thus the possibility of synthesizing it) but not the actual construction.

Structure of the paper. In Section 2, we introduce modal logic, refinement modal logic, action model logic, and arbitrary action model logic. In Section 3, we introduce relevant algebraic terminology. In Section 4, we present the methodology to define the algebraic semantics of dynamic epistemic logics. Finally, in Section 5, we present the algebraic semantics of refinement modal logic. Section 6 describes our results in view of prior works and concludes. The appendix contains the proofs of the results in Section 5.

2 Logical preliminaries

In this section, we succinctly introduce modal logic, action model logic [5], arbitrary action model logic [18], and refinement modal logic [23,7]. As all these logics are equally expressive, we can present them all as fragments of one logical language. Throughout the paper, we assume a non-empty, countable set of propositional atoms AtProp. We present here the single-agent version of these logics. All results in this section generalize to the multi-agent setting.

Models. A *(Kripke) frame* is a pair $\mathcal{F} = (S, R)$ where S is the *domain* consisting of *worlds* (or *states*), and $R \subseteq S \times S$ is a binary *accessibility relation*. Given $s \in S$, a pair (F, s), written as F_s, is a *pointed frame*, and a pair $(\mathcal{F}, T)$ with $T \subseteq S$ is a *multi-pointed frame* denoted $\mathcal{F}_T$. A *Kripke model* is a triple $M = (S, R, V)$ where (S, R) is a frame and where $V : \mathsf{AtProp} \to \mathcal{P}(S)$ is a *valuation* assigning to each propositional variable $p \in \mathsf{AtProp}$ the subset of the domain where the proposition p is true. Given a logical language $\mathcal{L}$, an *action model* over $\mathcal{L}$ is a triple $\alpha = (\mathsf{S}, \mathsf{R}, \mathsf{Pre})$ where (S, R) is a frame and where $\mathsf{Pre} : \mathsf{S} \to \mathcal{L}$ is a *precondition function*. The elements of the domain of an action model are called *actions*, or *action points*. Similarly to frames, we also define (multi-)pointed action models: given $\mathsf{s} \in \mathsf{S}$, a pair (α, u), written as α_u, is a *pointed action model*, and a pair (α, T) with $\mathsf{T} \subseteq \mathsf{S}$ is a *multi-pointed action model* and denoted as α_T. A (multi-)pointed action model is also called an *epistemic action*. The class of all action models *with finite domains* is $\mathcal{AM}$.

Let $M = (S, R, V)$ and $M' = (S', R', V')$ be given Kripke models. A non-empty relation $\mathfrak{R} \subseteq S \times S'$ is a *bisimulation* between M and M' if for all $(s, s') \in \mathfrak{R}$:

atoms $s \in V(p)$ iff $s' \in V'(p)$, for all $p \in \mathsf{AtProp}$;

forth $\forall t \in S$, if $R(s,t)$, $\exists t' \in S'$ such that $R'(s',t')$ and $(t,t') \in \mathfrak{R}$;

back $\forall t' \in S'$, if $R'(s',t')$, $\exists t \in S$ such that $R(s,t)$ and $(t,t') \in \mathfrak{R}$.

We write $M \simeq M'$ (M and M' *are bisimilar*) iff there is a bisimulation between M and M', and we write $M_s \simeq M'_{s'}$ (M_s and $M'_{s'}$ are bisimilar) iff this bisimulation links s and s'. A relation $\mathfrak{R}$ that satisfies **atoms**, **back** is called a *refinement*. We say that $M'_{s'}$ *refines* M_s and we write $M_s \succeq M'_{s'}$.

Languages. The language $\mathcal{L}_{\Box\forall\otimes\overline{\forall}}$ is inductively defined as:

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box\varphi \mid \forall\varphi \mid [\alpha_{\mathsf{u}}]\varphi \mid \overline{\forall}\varphi$$

where $p \in \mathsf{AtProp}$, $(\mathsf{S},\mathsf{R},\mathsf{Pre}) = \alpha \in \mathcal{AM}$ and $\mathsf{u} \in \mathsf{S}$.

We assume the usual abbreviations for propositional logical connectives, and also $\Diamond\varphi ::= \neg\Box\neg\varphi$, $[\alpha_{\mathsf{T}}]\varphi ::= \bigwedge_{\mathsf{u}\in\mathsf{T}}[\alpha_{\mathsf{u}}]\varphi$, $\langle\alpha_{\mathsf{u}}\rangle\varphi ::= \neg[\alpha_{\mathsf{u}}]\neg\varphi$ $\langle\alpha_{\mathsf{T}}\rangle\varphi ::= \neg[\alpha_{\mathsf{T}}]\neg\varphi$, $\exists\varphi ::= \neg\forall\neg\varphi$, and $\overline{\exists}\varphi ::= \neg\overline{\forall}\neg\varphi$.

The following fragments of the language will occur in the paper (with the obvious restrictions): $\mathcal{L}_{\Box}$ of modal logic (K); $\mathcal{L}_{\Box\otimes}$ of *action model logic* (AML); $\mathcal{L}_{\Box\otimes\overline{\forall}}$ of *arbitrary action model logic* (AAML); $\mathcal{L}_{\Box\forall}$ of *refinement modal logic* (RML).

Semantics. Let $M = (S,R,V)$ be a Kripke model, $s \in S$, $(\mathsf{S},\mathsf{R},\mathsf{Pre}) = \alpha \in \mathcal{AM}$ be an action model (note that it is finite), and $\mathsf{u} \in \mathsf{S}$. The interpretation of $\varphi \in \mathcal{L}_{\Box\forall\otimes\overline{\forall}}$ is defined inductively by

$$\begin{array}{lll}
M_s \models p & \text{iff} & s \in V(p)\\
M_s \models \varphi \wedge \psi & \text{iff} & M_s \models \varphi \text{ and } M_s \models \psi\\
M_s \models \neg\varphi & \text{iff} & M_s \not\models \varphi\\
M_s \models \Box\varphi & \text{iff} & \text{for all } t \in R(s) : M_t \models \varphi\\
M_s \models \forall\varphi & \text{iff} & \text{for all } M'_{s'} : M_s \succeq M'_{s'} \text{ implies } M'_{s'} \models \varphi\\
M_s \models [\alpha_{\mathsf{u}}]\varphi & \text{iff} & M_s \models \mathsf{Pre}(\mathsf{u}) \text{ implies } (M \otimes \alpha)_{(s,\mathsf{u})} \models \varphi\\
M_s \models \overline{\forall}\varphi & \text{iff} & \text{for all } \alpha_{\mathsf{u}} \in \mathcal{AM} : M_s \models [\alpha_{\mathsf{u}}]\varphi
\end{array}$$

where $M \otimes \alpha = (S^\alpha, R^\alpha, V^\alpha)$ is the *product update* defined as

$$\begin{array}{rcl}
S^\alpha & = & \{(s,\mathsf{u}) \in S \times \mathsf{S} \mid M_s \models \mathsf{Pre}(\mathsf{u})\}\\
(s,\mathsf{u})R^\alpha(s',\mathsf{u}') & \text{iff} & sRs' \text{ and } \mathsf{uRu}'\\
V^\alpha(p) & = & (V(p) \times \mathsf{S}) \cap S^\alpha
\end{array}$$

We say that $M \otimes \alpha$ is the model resulting from applying the epistemic action α on the model M. The *extension map* $[\![\cdot]\!]_M : \mathcal{L}_{\Box\forall\otimes\overline{\forall}} \to \mathcal{P}(S)$ for a $\varphi \in \mathcal{L}_{\Box\forall\otimes\overline{\forall}}$ is $[\![\varphi]\!]_M := \{s \in S \mid M_s \models \varphi\}$. A formula φ is *valid on* M, notation $M \models \varphi$, if for all $s \in S$, $M_s \models \varphi$. A formula φ is *valid*, if for all M, $M \models \varphi$. Instead of $(M \otimes \alpha)_{(s,\mathsf{u})}$ we may write $M_s \otimes \alpha_{\mathsf{u}}$.

Axiomatization AML. The axiomatization of AML consists of the rules and axioms of K [18, Definition IV.1] along with the following axioms and the rule of necessitation for dynamic box modalities:

AP $[\alpha_{\mathsf{u}}]p \leftrightarrow (\mathsf{Pre}(\mathsf{u}) \to p)$ for all $p \in \mathsf{AtProp}$
AN $[\alpha_{\mathsf{u}}]\neg\varphi \leftrightarrow (\mathsf{Pre}(\mathsf{u})\imath\neg[\alpha_{\mathsf{u}}]\varphi)$
AC $[\alpha_{\mathsf{u}}](\varphi \wedge \psi) \leftrightarrow ([\alpha_{\mathsf{u}}]\varphi \wedge [\alpha_{\mathsf{u}}]\psi)$
AK $[\alpha_{\mathsf{u}}]\Box\varphi \leftrightarrow (\mathsf{Pre}(\mathsf{u}) \to \bigwedge\{\Box[\alpha_{\mathsf{u'}}]\varphi \mid \mathsf{uRu'}\})$
AU $[\alpha_{\mathsf{T}}]\varphi \leftrightarrow \bigwedge_{\mathsf{u}\in\mathsf{T}}[\alpha_{\mathsf{u}}]\varphi$
NecA From φ infer $[\alpha_{\mathsf{u}}]\varphi$

Axiomatization RML**.** The axiomatization of RML consists of the rules and axioms of K and all substitution instances of the axioms and rules

R $\forall(\varphi \to \psi) \to \forall\varphi \to \forall\psi$
RProp $\forall p \leftrightarrow p$ and $\forall\neg p \leftrightarrow \neg p$
RK $\exists\nabla\Phi \leftrightarrow \bigwedge\Diamond\exists\Phi$
MP From $\varphi \to \psi$ and φ infer ψ
NecK From φ infer $\Box\varphi$
NecR From φ infer $\forall\varphi$

where for any finite set Φ of $\mathcal{L}_{\forall\Box}$ formulas we define by abbreviation $\nabla\Phi$ as $\Box\bigvee_{\varphi\in\Phi}\varphi \wedge \bigwedge_{\varphi\in\Phi}\Diamond\varphi$ and $\bigwedge\Diamond\exists\Phi$ as $\bigwedge_{\varphi\in\Phi}\Diamond\exists\varphi$, where $\bigvee_{\varphi\in\emptyset}\varphi := \bot$ and $\bigwedge_{\varphi\in\emptyset}\varphi := \top$.

Axiomatization AAML**.** The axiomatization of AAML is a substitution schema consisting of the rules and axioms of the logics RML and AML.

Some results about K**,** RML**,** AAML**,** AML**.**

- [23, Prop. 4&5]: The result of executing an epistemic action in a pointed model is a refinement of that model. Dually, for every refinement of a *finite* pointed model there is an epistemic action such that its execution results in a model bisimilar to that refinement.
- [18, Theorem V.3]: Let $\varphi \in \mathcal{L}_{\Box\forall\otimes\overline{\forall}}$. Then $\models \forall\varphi \leftrightarrow \overline{\forall}\varphi$.
- [5,7,18]: The logics K, RML, AAML, AML are all equally expressive.

Given a logic L, $\vdash \varphi$ means that φ is a theorem of the logic L, namely one can derive φ from the axioms and rules of L. In the Section 5, we use extensively the following theorem and lemma.

Theorem 2.1 *[18, Theorem V.3]: Let $\varphi \in \mathcal{L}_{\Box\otimes\forall}$. Then there exists a multi-pointed action model $\alpha^{\varphi}_{\mathsf{T}}$ such that $\vdash [\alpha^{\varphi}_{\mathsf{T}}]\varphi$ and $\vdash \langle\alpha^{\varphi}_{\mathsf{T}}\rangle\varphi \leftrightarrow \exists\varphi$.*

In [18] an algorithm is given to compute $\alpha^{\varphi}_{\mathsf{T}}$ from φ. In this construction a normal form is used that is called *cover disjunctive form* [20].

Lemma 2.2 *Let M_s be a pointed Kripke model and $\varphi \in \mathcal{L}_{\Box\otimes\forall}$. Then there exists a multi-pointed action model $\alpha^{\varphi}_{\mathsf{T}} = ((\mathsf{S}, \mathsf{R}, \mathsf{Pre}), \mathsf{T})$ such that $M_s \models \langle\alpha^{\varphi}_{\mathsf{T}}\rangle\varphi$ iff $M_s \models \exists\varphi$.*

Proof. Suppose that $M_s \models \exists\varphi$. Then by [18, Theorem V.3], there exists a multi-pointed action model $\alpha_{\mathsf{T}} = ((\mathsf{S}, \mathsf{R}, \mathsf{Pre}), \mathsf{T})$ such that $M_s \models \langle\alpha_{\mathsf{T}}\rangle\varphi$, which means $M_s \models \bigvee_{\mathsf{u}\in\mathsf{T}}\langle\alpha_{\mathsf{u}}\rangle\varphi$ with $\mathsf{T} \subseteq \mathsf{S}$. Therefore $M_s \models \langle\alpha_{\mathsf{S}}\rangle\varphi$.

For the other direction, suppose that there is a multi-pointed action model $\alpha_{\mathsf{T}} = ((\mathsf{S}, \mathsf{R}, \mathsf{Pre}), \mathsf{T})$ such that $M_s \models \langle \alpha_{\mathsf{T}} \rangle \varphi$. Then there exists $\mathsf{u} \in \mathsf{T}$ such that $M_s \models \langle \alpha_{\mathsf{u}} \rangle \varphi$, i.e. $M_s \otimes \alpha_{\mathsf{u}} \models \varphi$. □

3 Preliminaries on Algebras

In this section, we introduce relevant definitions and results on boolean algebras.

Boolean algebras. A *Boolean algebra* (**BA**) $\mathbb{A} = \langle A, \vee, \wedge, \neg, \bot, \top \rangle$ is an algebra with two binary operations $\vee$ (called 'join' or 'or') and $\wedge$ (called 'meet' or 'and'), one unary operation $\neg$ (called 'not' or 'complement'), and two nullary operations $\bot$ and $\top$ (called 'bottom' and 'top') which satisfy the following equations:

$a \wedge b = b \wedge a$	$a \vee b = b \vee a$	(commutativity)
$a \vee (b \vee c) = (a \vee b) \vee c$	$a \wedge (b \wedge c) = (a \wedge b) \wedge c$	(associativity)
$a \wedge (a \vee b) = a$	$a \vee (a \wedge b) = a$	(absorption)
$a \wedge \top = a$	$a \vee \bot = a$	(identity)
$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$	$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$	(distributivity)
$a \wedge \neg a = \bot$	$a \vee \neg a = \top$	(complementation)

Let X be a set and $\mathcal{P}(X)$ be the set of all the subsets of X. Denote with $\cup$, $\cap$ and $(-)^c$ the operations union, intersection and complement on $\mathcal{P}(X)$, respectively. Then $(\mathcal{P}(X), \cup, \cap, (-)^c, \emptyset, X)$ forms a **BA**.

Underlying poset of a boolean algebra. A **BA** $\mathbb{A} = (A, \vee, \wedge, \neg, \bot, \top)$ can also be seen as a poset (partially ordered set) $(A, \leq)$ where the order $\leq$ is defined as follows: $x \leq y$ iff $x \wedge y = x$ iff $x \vee y = y$, for any $x, y \in A$. We call $(A, \leq)$ the underlying poset of $\mathbb{A}$. Let $(A, \leq)$ be a poset, $a \in A$ and $S \subseteq A$, a is an *upper bound* (resp. *lower bound*) of S, if $s \leq a$ (resp. $a \leq s$) for every $s \in S$. The element $a \in A$ is the *least upper bound* of S if it is an upper bound of S and if $a \leq s$ for every upper bound s of S. The element $a \in \mathbb{A}$ is the *greatest lower bound* of S if it is a lower bound of S and if $s \leq a$ for every lower bound s of S. If they exist, the *least upper bound* of S is denoted by $\bigvee S$ and the greatest lower bound of S by $\bigwedge S$. For any **BA** $\mathbb{A} = \langle A, \vee, \wedge, \neg, \bot, \top \rangle$, $\bigvee S$ and $\bigwedge S$ of a finite subset $S \subseteq A$ always exist and are unique, however they may not exist if S is infinite.

Complete boolean algebras. A **BA** $\mathbb{A}$ is *complete* if $\bigvee S$ and $\bigwedge S$ exist for every $S \subseteq A$. The **BA** $(\mathcal{P}(X), \cup, \cap, (-)^c, \emptyset, X)$ is complete. The underlying order is given by the inclusion $\subseteq$, and for $S \subseteq \mathcal{P}(X)$, $\bigvee S$ and $\bigwedge S$ are respectively given by the union and the intersection. Formally, if I is an index set and $X_i \subseteq X$ for all $i \in I$, then $\bigvee_{i \in I} X_i = \bigcup_{i \in I} X_i$ and $\bigwedge_{i \in I} X_i = \bigcap_{i \in I} X_i$.

Boolean algebras with operators.

A *boolean algebra with operators* (**BAO**) is a structure $(\mathbb{A}, \{\Diamond_i\}_{i \in I})$ such that $\mathbb{A}$ is a **BA**, I is a non-empty finite set and $\Diamond_i : \mathbb{A} \to \mathbb{A}$ for every $i \in I$. A *normal* boolean algebra with operators is a **BAO** $(\mathbb{A}, \{\Diamond_i\}_{i\in I})$ such that the unary operations $\{\Diamond_i\}_{i\in I}$ on $\mathbb{A}$ satisfy $\Diamond_i \bot = \bot$ and $\Diamond_i(a \vee b) = \Diamond_i a \vee \Diamond_i b$, for all $a, b \in A$. The following definitions for boolean algebras are given for only one operator $\Diamond$. This can be done without loss of generality.

Congruence. A *congruence* θ on a **BAO** $\mathbb{A}$ is an equivalence relation which satisfies this compatibility property: for all $a, a', b, b' \in A$, if $a\theta b$ and $a'\theta b'$ then:

$$(\neg a)\theta(\neg b), \quad (a \vee a')\theta(b \vee b'), \quad (a \wedge a')\theta(b \wedge b') \quad \textit{and} \quad (\Diamond a)\theta(\Diamond b).$$

We denote by A/θ the set of the equivalence classes defined by the congruence θ, namely $A/\theta = \{[a]_\theta \mid a \in A\}$ with $[a]_\theta = \{b \in a \mid a\theta b\}$. There is a natural way to define the operations $\vee', \wedge', \neg'$ on the set A/θ of equivalence classes of $\mathbb{A}$ over θ. Namely, for all $a, b \in A$, we define

$$[a]_\theta \vee' [b]_\theta := [a \vee b]_\theta, \quad [a]_\theta \wedge' [b]_\theta := [a \wedge b]_\theta, \quad \textit{and} \quad \neg'[a]_\theta := [\neg a]_\theta.$$

It can be shown that $\mathbb{A}/\theta := \langle A/\theta, \vee', \wedge', \neg', [\bot]_\theta, [\top]_\theta \rangle$ is a **BA**. We call it the *quotient algebra* of $\mathbb{A}$ modulo θ.

Complex algebras. Let $\mathcal{F} := (S, R)$ be a Kripke frame. The *complex algebra* of $\mathcal{F}$, denoted $\mathcal{F}^+$, is the power set algebra $(\mathcal{P}(S), \cup, \cap, (-)^c, \emptyset, S)$ enriched with the operator $\Diamond_R : \mathcal{P}S \to \mathcal{P}S$ defined as $\Diamond_R(X) := \{s \in S \mid sRt$ for some $t \in X\} = R^{-1}[X]$ for every $X \in \mathcal{P}S$. We note that $\mathcal{F}^+$ is a normal **BAO**.

Complex algebras are the concrete **BAO**s that algebraize relational semantics [6, Theorem 5.25]. By means of complex algebras one can construct a **BAO** from a frame. For the other direction we need to construct the *ultrafilter frame* [6, Definition 5.34]. By transforming this frame in a complex algebra we get the *Jònson-Tarski Theorem* underlying the algebraization of modal logic:

Every **BAO** *can be embedded in the complex algebra of its ultrafilter frame [6, Theorem 5.43].*

Adjunction. A map $f : (A, \leq_A) \to (B, \leq_B)$ between two posets is *monotone* if $a \leq_A b$ implies $f(a) \leq_B f(b)$ for all $a, b \in A$. A pair (f, g) of monotone maps $f : \mathbb{A} \to \mathbb{B}$ and $g : \mathbb{B} \to \mathbb{A}$ between two posets forms an *adjunction* (denoted $f \dashv g$) between $\mathbb{A}$ and $\mathbb{B}$ if $f(a) \leq_B b$ is equivalent to $a \leq_A g(b)$, for all $a \in A$ and $b \in B$. If $f \dashv g$, then g is a *right adjoint* and f a *left adjoint*.

Let $(\mathbb{A}, \Diamond)$ be a complex algebra of some Kripke frame, and $\Box := \neg\Diamond\neg$, then $\Diamond$ is a left adjoint and $\Box$ is a right adjoint. Moreover, there exist $\blacklozenge$ and $\blacksquare$ such that $\Diamond \dashv \blacksquare$ and $\blacklozenge \dashv \Box$.

Algebraic models.

An *algebraic model* is a tuple $\mathcal{A} = (\mathbb{A}, V)$ such that $\mathbb{A}$ is a normal **BAO** and $V : \mathsf{AtProp} \to \mathbb{A}$. Let $M := (\mathcal{F}, V)$ with $V : \mathsf{AtProp} \to \mathcal{P}S$ be a Kripke model, the *algebraic model associated with* M is the tuple $\mathcal{A} = (\mathcal{F}^+, V)$ where $\mathcal{F}^+$ is the complex algebra of $\mathcal{F}$. Notice that the valuation V (resp. the extension map $[\![\cdot]\!]_M$) sends atomic propositions (resp. formulas) to elements in $\mathbb{A}$.

We will rely on the duality between Kripke frames and normal boolean algebras with operators to define the algebraic semantics of arbitrary action model logic and refinement modal logic.

4 Algebraic semantics of Arbitrary Action Model Logic

In this section, first we present the methodology to define epistemic updates on algebras (Section 4.1), and then we give the algebraic semantics of Action Model Logic (Section 4.2). Sections 4.1 and 4.2 report on results introduced in [21,22].

4.1 Epistemic update on algebras

We first describe the methodology to define the epistemic update on normal boolean algebras with operators, and then the mathematical steps to compute the updated algebra.

Methodology. Let $M = (S, R, V)$ be a Kripke model and $\alpha = (\mathsf{S}, \mathsf{R}, \mathsf{Pre})$ be an action model over $\mathcal{L}_{\Box\otimes}$. The product update $M \otimes \alpha$ defined in Section 2 can be built in an algebraic way in two steps as follows.

<u>Step 1.</u> We define the following intermediate model

$$\coprod_{\alpha} M = (\coprod_{\mathsf{S}} S, R \times \mathsf{R}, \coprod_{\alpha} V)$$

where

(i) $\coprod_{\mathsf{S}} S \simeq S \times \mathsf{S}$ is the $|\mathsf{S}|$-fold coproduct of S, which is set-isomorphic to cartesian product of $S \times \mathsf{S}$,

(ii) $R \times \mathsf{R}$ is the binary relation on $\coprod_{\mathsf{S}} S$ defined as

$$(s, \mathsf{u})(R \times \mathsf{R})(s', \mathsf{u}') \quad \text{iff} \quad sRs' \text{ and } \mathsf{u}\mathsf{R}\mathsf{u}',$$

(iii) $\coprod_{\alpha} V : \mathsf{AtProp} \to \mathcal{P}(\coprod_{\mathsf{S}} S)$ such that for every $p \in \mathsf{AtProp}$

$$\coprod_{\alpha} V(p) = \coprod_{\alpha} (V(p)) = V(p) \times \mathsf{S}.$$

<u>Step 2.</u> $M \otimes \alpha$ is the submodel of $\coprod_{\alpha} M$ that contains the tuples $(s, u) \in \coprod_{\mathsf{S}} S$ such that $M_s \models \mathsf{Pre}(u)$.

This two-step-account of the product update construction can be seen as a pseudo-coproduct, as illustrated by the following diagram

$$M \hookrightarrow \coprod_{\alpha} M \hookleftarrow M \otimes \alpha.$$

This perspective makes it possible to use the duality between products and coproducts in category theory (cf. [10,1]): coproducts can be dually characterized as products, and subobjects as quotients. Using this result, the update of M

with the action model α, regarded as a "subobject after coproduct" concatenation, can be dually characterized on its algebraic counterpart $(\mathbb{A}, V)$ by means of a "quotient after product" concatenation, as illustrated in the following diagram:

$$\mathbb{A} \leftarrowtail \prod_{\alpha} \mathbb{A} \twoheadrightarrow \mathbb{A}^{\alpha}.$$

Indeed, the pseudo-coproduct $\coprod_{\alpha} M$ is dually characterized as a *pseudo-product* $\prod_{\alpha} \mathbb{A}$ and an appropriate *quotient* of $\prod_{\alpha} \mathbb{A}$ is then taken to dually characterize the submodel step. This construction we now define.

Product on Sets. Recall that, in the category of sets, the *product* is the Cartesian product. Namely, given a family of sets $(X_i)_{i\in I}$, as $\prod_{i\in I} X_i := \{(x_i)_{i\in I} \mid \forall i \in I, x_i \in X_i\}$ with the canonical projections $\pi_j : \prod_{i\in I} X_i \to X_j$ defined as $\pi_j((x_i)_{i\in I}) := x_j$.

Dual characterization of the intermediate structure.

Definition 4.1 [Action model on algebras] For every algebra $\mathbb{A}$, we define an *action model over* $\mathbb{A}$ as a tuple $a = (\mathsf{S}, \mathsf{R}, \mathsf{Pre}_a)$ such that S is a finite nonempty set, $\mathsf{R} \subseteq \mathsf{S} \times \mathsf{S}$ and $\mathsf{Pre}_a : \mathsf{S} \to \mathbb{A}$. As for Kripke models, one can define pointed action models (a, u) over $\mathbb{A}$ with $\mathsf{u} \in a$ denoted a_{u}.

Clearly, for every Kripke model $M = (S, R, V)$, each action model $\alpha = (\mathsf{S}, \mathsf{R}, \mathsf{Pre})$ over $\mathcal{L}_{\Box\otimes}$ induces a corresponding action model a over the complex algebra $\mathbb{A}$ of the underlying frame (S, R) of M, via the valuation $V : \mathsf{AtProp} \to \mathbb{A}$, namely, a is defined as $a = (\mathsf{S}, \mathsf{R}, \mathsf{Pre}_a)$, with $\mathsf{Pre}_a = V \circ \mathsf{Pre}_a$.

For every **BA** $\mathbb{A}$ and every action model $a = (\mathsf{S}, \mathsf{R}, \mathsf{Pre}_a)$ over $\mathbb{A}$, let $\prod_a \mathbb{A}$ be the $|\mathsf{S}|$-fold product of $\mathbb{A}$, which is set-isomorphic to the collection $\mathbb{A}^{\mathsf{S}}$ of the set maps $f : \mathsf{S} \to \mathbb{A}$. The set $\mathbb{A}^{\mathsf{S}}$ can be canonically endowed with the same algebraic structure as $\mathbb{A}$ by pointwise lifting the operations on $\mathbb{A}$ [4]; as such, it satisfies the same equations as $\mathbb{A}$.

Definition 4.2 Let $(\mathbb{A}, \Diamond)$ be a normal **BAO**, $\Box := \neg\Diamond\neg$ and $a = (\mathsf{S}, \mathsf{R}, \mathsf{Pre}_a)$ be an action model over $\mathbb{A}$, we define the operations $\Diamond^{\prod_a \mathbb{A}}$ and $\Box^{\prod_a \mathbb{A}}$ on the product $\prod_a \mathbb{A}$ as follows: for every $f : \mathsf{S} \to \mathbb{A}$,

$$\begin{aligned} \Diamond^{\prod_a \mathbb{A}} f : \mathsf{S} &\to \mathbb{A} & \qquad \Box^{\prod_a \mathbb{A}} f : \mathsf{S} &\to \mathbb{A} \\ \mathsf{u} &\mapsto \bigvee\{\Diamond^{\mathbb{A}} f(\mathsf{u}') \mid \mathsf{u}\mathsf{R}\mathsf{u}'\} & \mathsf{u} &\mapsto \bigwedge\{\Box^{\mathbb{A}} f(\mathsf{u}') \mid \mathsf{u}\mathsf{R}\mathsf{u}'\}. \end{aligned}$$

The operators $\Diamond^{\prod_a \mathbb{A}}$ and $\Box^{\prod_a \mathbb{A}}$ are normal modal operators such that $\Box^{\prod_a \mathbb{A}} = \neg\Diamond^{\prod_a \mathbb{A}}\neg$, and the product algebra $(\prod_a \mathbb{A}, \Diamond^{\prod_a \mathbb{A}})$ is a normal **BAO** [21, Proposition 3.2]. Also, if $\mathbb{A}$ is the complex algebra of the underlying frame of the Kripke model $(\mathcal{F}, V)$ and if the action model a over $\mathbb{A}$ is derived from the action model $\alpha = (\mathsf{S}, \mathsf{R}, \mathsf{Pre})$ over $\mathcal{L}_{\Box\otimes}$, then $(\prod_a \mathbb{A}, \Diamond^{\prod_a \mathbb{A}})$ is isomorphic to

[4] For all $f, g : \mathsf{S} \to \mathbb{A}$, the maps $(f \wedge^{\prod_a \mathbb{A}} g), \neg^{\prod_a \mathbb{A}} f \bot^{\prod_a \mathbb{A}} : \mathsf{S} \to \mathbb{A}$ are respectively defined as follows: $(f \wedge^{\prod_a \mathbb{A}} g)(\mathsf{u}) := f(\mathsf{u}) \wedge^{\mathbb{A}} g(\mathsf{u})$, $(\neg^{\prod_a \mathbb{A}} f)(\mathsf{u}) := \neg^{\mathbb{A}} f(\mathsf{u})$ and $\bot^{\prod_a \mathbb{A}}(\mathsf{u}) := \bot$.

the complex algebra of the underlying frame $\coprod_a \mathcal{F}$ of the intermediate model $\coprod_a M$ [21, Proposition 3.1].

Quotient of the intermediate structure. Let $\mathbb{A}$ be a normal **BAO** and $a = (\mathsf{S}, \mathsf{R}, \mathsf{Pre}_a)$ be an action model over $\mathbb{A}$. The equivalence relation $\equiv_a$ on $\prod_a \mathbb{A}$ is defined as follows: for all $f, g \in \mathbb{A}^{\mathsf{S}}$,

$$f \equiv_a g \quad \text{iff} \quad f \wedge \mathsf{Pre}_a = g \wedge \mathsf{Pre}_a.$$

For any $f \in \mathbb{A}^{\mathsf{S}}$, we denote by $[f]_a$ its equivalence class. The subscript will be dropped whenever it causes no confusion. Let the quotient algebra $\mathbb{A}^{\mathsf{S}}/{\equiv_a}$ be denoted by $\mathbb{A}^a$. This quotient is compatible with the boolean operations, however it is not compatible with the modal operators, indeed $f \equiv g$ does not imply that $\Diamond f \equiv \Diamond g$. So we need to choose a definition for the modalities on $\mathbb{A}^a$: let for every $f \in \mathbb{A}^{\mathsf{S}}$,

$$\Diamond^a[f] := [\Diamond^{\prod_a \mathbb{A}}(f \wedge \mathsf{Pre}_a)] \qquad \text{and} \qquad \Box^a[f] := [\Box^{\prod_a \mathbb{A}}(f \rightarrow \mathsf{Pre}_a)].$$

The operators $\Diamond^a$ and $\Box^a$ are normal modal operators such that $\Box^a = \neg\Diamond^a\neg$ and $(\mathbb{A}^a, \Diamond^a)$ is a normal **BAO**. Moreover, if $(\mathbb{A}, V) = (\mathcal{F}^+, V)$ for some Kripke model $M = (\mathcal{F}, V)$, and if the action model a over $\mathbb{A}$ is derived from some action model $\alpha = (\mathsf{S}, \mathsf{R}, \mathsf{Pre})$ over $\mathcal{L}_{\Box\otimes}$, then $\mathbb{A}^a \cong_{\mathbf{BAO}} \mathcal{F}^{\alpha+}$, in which $\mathcal{F}^\alpha$ is the underlying frame of the updated model $M \otimes \alpha$.

Definition 4.3 Let $(\mathbb{A}, \Diamond)$ be a normal **BAO** and $a = (\mathsf{S}, \mathsf{R}, \mathsf{Pre}_a)$ be an action model over $\mathbb{A}$. The update of $\mathbb{A}$ with a is $\mathbb{A}^a := (\mathbb{A}^{\mathsf{S}}/\equiv_a, \Diamond^a)$, where $(\mathbb{A}^{\mathsf{S}}/\equiv_a)$ is the quotient algebra and $\Diamond^a$ is the normal modality, as above.

4.2 Algebraic semantics of action model logic

In this section we report on the algebraic semantics of action model logic proposed by [21,22]. We recall that that an *algebraic model* is a tuple $\mathcal{A} = (\mathbb{A}, V)$ such that $\mathbb{A}$ is a normal **BAO** and $V : \mathsf{AtProp} \rightarrow \mathbb{A}$.

Definition 4.4 Let $\mathcal{A} = (\mathbb{A}, V)$ be an algebraic model, $\alpha = (\mathsf{S}, \mathsf{R}, \mathsf{Pre}_\alpha)$ an action model over $\mathcal{L}_{\Box\otimes}$ and $a = (\mathsf{S}, \mathsf{R}, \mathsf{Pre}_a)$ the action model induced by α via V.
The *intermediate algebraic model* $\prod_\alpha \mathcal{A}$ is defined as

$$\prod_\alpha \mathcal{A} := (\prod_a \mathbb{A}, \prod_a V)$$

where, for any $p \in \mathsf{AtProp}$, the map $(\prod_a V)(p) : \mathsf{S} \rightarrow \prod_a \mathbb{A}$ is such that $((\prod_a V)(p))\,(\mathsf{u}) = V(p)$.
The *updated algebraic model* $\mathcal{A}^\alpha$ is defined as

$$\mathcal{A}^\alpha := (\mathbb{A}^a, V^a)$$

where $V^a : \mathsf{AtProp} \rightarrow \mathbb{A}^a$ is the map such that $V^a(p) = [\prod_a V(p)]_a$.

Let $\mathcal{A} = (\mathbb{A}, V)$ be an algebraic model, $\alpha = (\mathsf{S}, \mathsf{R}, \mathsf{Pre}_\alpha)$ be an action model over $\mathcal{L}_{\Box\otimes}$, and a the action model induced by α via V. Let $\pi_\mathsf{u} : \prod_a \mathbb{A} \to \mathbb{A}$ be the projection on the u-indexed coordinate that maps every $f \in \prod_a \mathbb{A}$ to $f(\mathsf{u})$, and let $i' : \mathbb{A}^a \to \prod_a \mathbb{A}$ be defined as $i'([f]) = f \wedge \mathsf{Pre}_a$ for all $[f] \in \mathbb{A}^a$. Then:

Definition 4.5 For every algebraic model $\mathcal{A} = (\mathbb{A}, V)$, its extension map $[\![.]\!]_\mathcal{A} : \mathcal{L}_{\Box\otimes} \to \mathbb{A}$ is defined recursively as

$$\begin{aligned}
[\![p]\!]_\mathcal{A} &:= v(p) \\
[\![\bot]\!]_\mathcal{A} &:= \bot^\mathbb{A} \\
[\![\circ\varphi]\!]_\mathcal{A} &:= \circ^\mathbb{A}[\![\varphi]\!]_\mathcal{A} && \text{for } \circ \in \{\neg, \Diamond, \Box\} \\
[\![\varphi \bullet \psi]\!]_\mathcal{A} &:= [\![\varphi]\!]_\mathcal{A} \bullet^\mathbb{A} [\![\psi]\!]_\mathcal{A} && \text{for } \bullet \in \{\vee, \wedge, \to\} \\
[\![\langle\alpha_\mathsf{u}\rangle\varphi]\!]_\mathcal{A} &:= [\![\mathsf{Pre}(\mathsf{u})]\!]_\mathcal{A} \wedge^\mathbb{A} \pi_\mathsf{u} \circ i'([\![\varphi]\!]_{\mathcal{A}^\alpha}) \\
[\![[\alpha_\mathsf{u}]\varphi]\!]_\mathcal{A} &:= [\![\mathsf{Pre}(\mathsf{u})]\!]_\mathcal{A} \to^\mathbb{A} \pi_\mathsf{u} \circ i'([\![\varphi]\!]_{\mathcal{A}^\alpha})
\end{aligned}$$

5 Algebraic semantics of refinement modal logic

In this section, we present our main result, namely an algebraic semantics for RML. First we introduce the notion of *refinement morphism*. It is the analogue on normal boolean algebras with operators of the notion of refinement between Kripke models. Then we define *refinement algebra*. This is used in the definition of the *algebraic semantics of refinement modal logic*. Finally we prove that RML is sound and complete w.r.t. this semantics.

Throughout this section we adopt the following notational conventions.

- $\mathbb{A}$ is a complete normal **BAO**;
- $\mathcal{A} = (\mathbb{A}, V)$ is an algebraic model;
- $\alpha^\varphi_\mathsf{S} = (\mathsf{S}, \mathsf{R}, \mathsf{Pre}^\varphi)$ denotes the multi-pointed action model obtained by using Hales algorithm on the formula φ (cf. Lemma 2.2 and [18, Lemmas V.I and V.II]);
- $a^\varphi = (\mathsf{S}, \mathsf{R}, \mathsf{Pre}_a)$ denotes the action model over $\mathbb{A}$ induced by $\alpha^\varphi_\mathsf{S}$ via V;
- $\mathcal{A}^\varphi$ denotes the algebraic model $\mathcal{A}^{\alpha^\varphi_\mathsf{S}}$ (Definition 4.4);
- $\mathbb{A}^\varphi$ denotes the **BAO** underlying the algebraic model $\mathcal{A}^\varphi$;
- $[\![.]\!]_\mathcal{A} : \varphi \in \mathcal{L}_{\Box\forall\otimes\overline{\forall}} \to \mathbb{A}$ is the *extension map* of V on $\mathcal{A}$ such that $[\![\varphi]\!]_\mathcal{A}$ follows Definition 4.5 for all logical connectives except quantifiers and such that

$$\begin{aligned}
[\![\overline{\exists}\varphi]\!]_\mathcal{A} = [\![\exists\varphi]\!]_\mathcal{A} &:= \bigvee_{\mathsf{u}\in\mathsf{S}}[\![\langle\alpha^\varphi_\mathsf{u}\rangle\varphi]\!]_\mathcal{A} = \bigvee_{\mathsf{u}\in\mathsf{S}}([\![\mathsf{Pre}^\varphi(\mathsf{u})]\!]_\mathcal{A} \wedge \pi_\mathsf{u} \circ i'([\![\varphi]\!]_{\mathcal{A}^\varphi})) \\
[\![\forall\varphi]\!]_\mathcal{A} &:= [\![\neg\exists\neg\varphi]\!]_\mathcal{A} \\
[\![\overline{\forall}\varphi]\!]_\mathcal{A} &:= [\![\neg\overline{\exists}\neg\varphi]\!]_\mathcal{A}
\end{aligned}$$

Refinement morphisms and their adjoints. The dual notion of refinement on algebras is the *refinement morphism*. We prove that refinement morphisms are right adjoints.

Definition 5.1 Let $\mathbb{A}$ and $\mathbb{A}'$ be two normal **BAO**s. A map $f : \mathbb{A} \to \mathbb{A}'$ is a *refinement morphism* if it is monotone, preserves $\bot$ and $\vee$, and satisfies the inequality $\blacklozenge^{\mathbb{A}'} \circ f^{\varphi} \leq f^{\varphi} \circ \blacklozenge^{\mathbb{A}}$ where $\blacklozenge \dashv \Box$ (cf. page 44).

The inequality $\blacklozenge^{\mathbb{A}'} \circ f^{\varphi} \leq f^{\varphi} \circ \blacklozenge^{\mathbb{A}}$ is the dual notion on algebras of the back condition in the refinement relation (cf. page 41).

Definition 5.2 For any algebraic model $\mathcal{A} = (\mathbb{A}, V)$ and any formula $\varphi \in \mathcal{L}_{\otimes\forall}$, we define the maps f^{φ} and g^{φ} as follows:

$$\begin{aligned} f^{\varphi} : \mathbb{A} &\to \mathbb{A}^{\varphi} & \qquad g^{\varphi} : \mathbb{A}^{\varphi} &\to \mathbb{A} \\ b &\mapsto [f_b] & [h] &\mapsto \bigvee_{\mathsf{u} \in \mathsf{S}} (h(\mathsf{u}) \wedge \mathsf{Pre}_a(\mathsf{u})) \end{aligned}$$

where $f_b : \mathsf{S} \to \mathbb{A}$ is the map such that $f_b(\mathsf{u}) := b \wedge \mathsf{Pre}_a(\mathsf{u})$ and $a = (\mathsf{S}, \mathsf{R}, \mathsf{Pre}_a)$ is the action model induced by $\alpha^{\varphi}_{\mathsf{S}}$ via V.

Lemma 5.3 *For any algebraic model $\mathcal{A}$ and any formula $\varphi \in \mathcal{L}_{\otimes\forall}$,*

(i) *the map f^{φ} is a refinement morphism,*

(ii) *the map g^{φ} is monotone and preserves arbitrary joins,*

(iii) $g^{\varphi} \dashv f^{\varphi}$.

Lemma 5.4 *Let $M = (S, R, V)$ and $M' = (S', R', V')$ be Kripke models, and let, respectively, $\mathcal{A}$ and $\mathcal{A}'$ be their algebraic models.*

(There exists a refinement morphism $f : \mathbb{A} \to \mathbb{A}'$) iff $M_s \succeq M'_{s'}$.

Lemma 5.5 *For any algebraic model $\mathcal{A}$ and any formula $\varphi \in \mathcal{L}_{\otimes\forall}$,*

$$[\![\exists\varphi]\!]_{\mathcal{A}} = g^{\varphi}([\![\varphi]\!]_{\mathcal{A}^{\varphi}}). \tag{1}$$

Refinement. We aim at proposing an algebraic semantics for the refinement modality $\exists$, i.e. for any algebraic model $\mathcal{A} = (\mathbb{A}, V)$, we want to find a normal **BAO** $\mathfrak{A}_{\mathcal{A}}$ and a map $G : \mathfrak{A}_{\mathcal{A}} \to \mathbb{A}$ such that for any $\varphi \in \mathcal{L}_{\Box\forall}$,

$$[\![\exists\varphi]\!]_{\mathcal{A}} = G([\![\varphi]\!]_{\mathfrak{A}_{\mathcal{A}}}).$$

To do so, we introduce a normal **BAO** $\mathfrak{A}_{\mathcal{A}}$ such that $\mathbb{A}^{\varphi}$ is a subalgebra of $\mathfrak{A}_{\mathcal{A}}$ for any $\varphi \in \mathcal{L}_{\Box\forall}$. Hence, for every algebraic model $\mathcal{A} = (\mathbb{A}, V)$, we define the following algebraic structure:

$$\mathfrak{A}_{\mathcal{A}} := \prod_{\varphi \in \mathcal{L}_{\Box\forall}} \mathbb{A}^{\varphi}.$$

Elements of $\mathfrak{A}_{\mathcal{A}}$ are tuples $(b^{\varphi})_{\varphi \in \mathcal{L}_{\Box\forall}}$ where $b^{\varphi} \in \mathbb{A}^{\varphi}$. When there is no risk of confusion, we write $(b^{\varphi})_{\varphi}$ instead of $(b^{\varphi})_{\varphi \in \mathcal{L}_{\Box\forall}}$ and $\mathfrak{A}$ instead of $\mathfrak{A}_{\mathcal{A}}$.

Recall that the product of any family $\{\mathbb{A}_i\}_{i \in I}$ of normal **BAO**s, where I may be an uncountable set, is a normal **BAO** [8, Section 7]. The operations on the algebra

$$\mathfrak{A} = \left(\prod_{\varphi \in \mathcal{L}_{\Box\forall}} \mathbb{A}^{\varphi}, \vee^{\mathfrak{A}}, \wedge^{\mathfrak{A}}, \neg^{\mathfrak{A}}, \bot^{\mathfrak{A}}, \top^{\mathfrak{A}}, \Diamond^{\mathfrak{A}} \right),$$

are the following, where $\Box^{\mathfrak{A}} := \neg^{\mathfrak{A}} \Diamond^{\mathfrak{A}} \neg^{\mathfrak{A}}$. For all $(b^{\varphi})_{\varphi}, (c^{\varphi})_{\varphi} \in \mathfrak{A}$,

Constants
$\bot^{\mathfrak{A}} = (\bot^{\varphi})_{\varphi}$, $\top^{\mathfrak{A}} = (\top^{\varphi})_{\varphi}$

Negation
$\neg^{\mathfrak{A}}(b^{\varphi})_{\varphi} = (\neg b^{\varphi})_{\varphi}$

Join and meet
$(b^{\varphi})_{\varphi} \vee^{\mathfrak{A}} (c^{\varphi})_{\varphi} = (b^{\varphi} \vee c^{\varphi})_{\varphi}$
$(b^{\varphi})_{\varphi} \wedge^{\mathfrak{A}} (c^{\varphi})_{\varphi} = (b^{\varphi} \wedge c^{\varphi})_{\varphi}$

Modal operators
$\Diamond^{\mathfrak{A}}(b^{\varphi})_{\varphi} = (\Diamond^{\mathbb{A}^{\varphi}} b^{\varphi})_{\varphi}$
$\Box^{\mathfrak{A}}(b^{\varphi})_{\varphi} = (\Box^{\mathbb{A}^{\varphi}} b^{\varphi})_{\varphi}$

One can easily verify that $\mathfrak{A}$ is a normal boolean algebra. We call $\mathfrak{A}$ the *refinement algebra* of the algebraic model $\mathcal{A} = (\mathbb{A}, V)$. One can also define the modal operators $\blacklozenge^{\mathfrak{A}}$ and $\blacksquare^{\mathfrak{A}}$ as $\blacklozenge^{\mathfrak{A}}(b^{\varphi})_{\varphi} = \left(\blacklozenge^{\mathbb{A}^{\varphi}} b^{\varphi}\right)_{\varphi}$ and $\blacksquare^{\mathfrak{A}}(b^{\varphi})_{\varphi} = \left(\blacksquare^{\mathbb{A}^{\varphi}} b^{\varphi}\right)_{\varphi}$, for any $(b^{\varphi})_{\varphi} \in \mathfrak{A}$ such that $\blacklozenge^{\mathfrak{A}} \dashv \Box^{\mathfrak{A}}$ and $\Diamond^{\mathfrak{A}} \dashv \blacksquare^{\mathfrak{A}}$.

Definition 5.6 For every algebraic model $\mathcal{A} = (\mathbb{A}, V)$, the maps $F_{\mathcal{A}}$ and $G_{\mathcal{A}}$ are defined as

$$F_{\mathcal{A}} : \mathbb{A} \to \mathfrak{A}_{\mathcal{A}}, \quad a \mapsto \prod_{\varphi \in \mathcal{L}_{\Box\forall}} (f^{\varphi}(a)) \qquad G_{\mathcal{A}} : \mathfrak{A}_{\mathcal{A}} \to \mathbb{A}, \quad ([h]^{\varphi})_{\varphi} \mapsto \bigvee_{\varphi} g^{\varphi}([h]^{\varphi})$$

where $f^{\varphi} : \mathbb{A} \to \mathbb{A}^{\varphi}$ and $g^{\varphi} : \mathbb{A}^{\varphi} \to \mathbb{A}$ are the maps of Def. 5.2.

Lemma 5.7 *$\mathcal{A} = (\mathbb{A}, V)$ be an algebraic model and $\mathfrak{A}_{\mathcal{A}}$ its refinement algebra. Then*

(i) *the map $F_{\mathcal{A}}$ is a refinement morphism,*

(ii) *the map $G_{\mathcal{A}}$ is monotone and preserves $\bot$, $\top$ and finite joins,*

(iii) $G_{\mathcal{A}} \dashv F_{\mathcal{A}}$.

Unless confusion results we write F for $F_{\mathcal{A}}$ and G for $G_{\mathcal{A}}$.

Algebraic semantics of refinement modal logic. We now conclude with the algebraic semantics of refinement modal logic and the corresponding completeness result.

Definition 5.8 Let $\mathcal{A} = (\mathbb{A}, V)$ be an algebraic model and $\mathfrak{A}$ its refinement algebra. Let $\mathcal{A}'$ be the algebraic model $(\mathfrak{A}, \mathcal{V})$ with $\mathcal{V} : \mathsf{AtProp} \to \mathfrak{A}$ and $\mathcal{V}(p) = (F \circ V)(p)$. The extension map $[\![.]\!]'_{\mathcal{A}} : \mathcal{L}_{\Box\forall} \to \mathbb{A}$ is defined as follows.

$$\begin{aligned}
[\![p]\!]'_{\mathcal{A}} &:= V(p) \\
[\![\bot]\!]'_{\mathcal{A}} &:= \bot^{\mathbb{A}} \\
[\![\circ\varphi]\!]'_{\mathcal{A}} &:= \circ^{\mathbb{A}}[\![\varphi]\!]'_{\mathcal{A}} && \text{for } \circ \in \{\neg, \Diamond, \Box\} \\
[\![\varphi \bullet \psi]\!]'_{\mathcal{A}} &:= [\![\varphi]\!]'_{\mathcal{A}} \bullet^{\mathbb{A}} [\![\psi]\!]'_{\mathcal{A}} && \text{for } \bullet \in \{\vee, \wedge, \to\} \\
[\![\exists\varphi]\!]'_{\mathcal{A}} &:= G([\![\varphi]\!]'_{\mathcal{A}'})
\end{aligned}$$

Theorem 5.9 *The axiomatization of* RML *(cf. page 42) is sound and complete with respect to the algebraic semantics defined above.*

6 Conclusion and further research

We have proposed an algebraic semantics for refinement modal logic. Using action model synthesis and the algebraic characterization of epistemic updates, we have introduced the abstract notion of refinement on normal boolean algebras, and showed the soundness and completeness of refinement modal logic with respect to this algebraic semantics.

Our methodology builds on and further develops recent work [21,22] applying duality theory to dynamic epistemic logic. As part of this research program, proof systems for intuitionistic AML have been introduced [17,15], and gave rise to the novel methodology of multi-type display calculi [14], which has been applied not only to AML [13], but also to propositional dynamic logic [12] and inquisitive logic [16]. A natural direction is to pursue this research program also on refinement modal logic. We plan to weaken the classical propositional modal logical base to a non-classical propositional modal logical base, and to develop multi-type calculi for such non-classical modal logics with refinement quantiers, for example refinement intuitionistic (modal) logic.

An other step to take would be to generalize the algebraic semantics of RML to the multi-agent framework. In this framework, the refinement modality $\exists$ is indexed by a agent, hence we have modalities $\{\exists_i\}_{i\in\mathsf{Ag}}$ where Ag is the set of agents. The only difficulty to generalize our result is to prove, algebraically, the soundness of the additional axioms:

$$\exists_i \nabla_j \Phi \leftrightarrow \nabla_j \{\exists_i \varphi\}_{\varphi\in\Phi} \quad \text{where } i \neq j$$
$$\exists_i \bigwedge_{j\in J} \nabla_j \Phi^j \leftrightarrow \bigwedge_{j\in J} \exists_i \nabla_j \Phi^j \quad \text{where } J \subseteq \mathsf{Ag}.$$

Indeed, as the reader can see in the appendix, the soundness proofs can be quite involved. This step is however very useful to propose a good proof system for the multi-agent refinement modal logic.

Appendix

A Proofs Section 5

This section contains the proof of lemmas and theorems from section 5.

Proof of Lemma 5.3. (i). That f^φ is monotone and preserves $\bot$ and $\vee$ follows from [21, Fact 11.4]. That $(\blacklozenge^{\mathbb{A}^\varphi} \circ f^\varphi) \leq (f^\varphi \circ \blacklozenge^{\mathbb{A}})$ follows from the following chain of inequalities. For every $b \in \mathbb{A}$ and every $\mathsf{u} \in \mathsf{S}$,

$$
\begin{aligned}
&(\blacklozenge^{\mathbb{A}^\varphi} \circ f^\varphi(b))(\mathsf{u}) = \blacklozenge^{\mathbb{A}^\varphi}([f_b])(\mathsf{u}) = [\blacklozenge^{\Pi_\alpha \mathbb{A}}(f_b \wedge \mathsf{Pre}_a)](\mathsf{u}) \\
&= \left(\blacklozenge^{\Pi_\alpha \mathbb{A}}(f_b \wedge \mathsf{Pre}_a) \wedge \mathsf{Pre}_a\right)(\mathsf{u}) = \left(\blacklozenge^{\Pi_\alpha \mathbb{A}}(f_b \wedge \mathsf{Pre}_a)(\mathsf{u})\right) \wedge \mathsf{Pre}_a(\mathsf{u}) \\
&= \bigvee \{\blacklozenge^{\mathbb{A}}(f_b \wedge \mathsf{Pre}_a)(\mathsf{t}) \mid \mathsf{uRt}\} \wedge \mathsf{Pre}_a(\mathsf{u}) \\
&\leq \bigvee \{\blacklozenge^{\mathbb{A}} f_b(\mathsf{t}) \wedge \blacklozenge^{\mathbb{A}}\mathsf{Pre}_a(\mathsf{t}) \mid \mathsf{tRu}\} \wedge \mathsf{Pre}_a(\mathsf{u}) \\
&= \bigvee \{(\blacklozenge^{\mathbb{A}}(b \wedge \mathsf{Pre}_a(\mathsf{t})) \wedge \blacklozenge^{\mathbb{A}}\mathsf{Pre}_a(\mathsf{t})) \mid \mathsf{tRu}\} \wedge \mathsf{Pre}_a(\mathsf{u}) \\
&\leq \bigvee \{\blacklozenge^{\mathbb{A}} b \wedge \blacklozenge \mathsf{Pre}_a(\mathsf{t}) \wedge \blacklozenge^{\mathbb{A}}\mathsf{Pre}_a(\mathsf{t}) \mid \mathsf{tRu}\} \wedge \mathsf{Pre}_a(\mathsf{u}) \\
&\leq \bigvee \{\blacklozenge^{\mathbb{A}} b \wedge \blacklozenge^{\mathbb{A}}\mathsf{Pre}_a(\mathsf{t}) \mid \mathsf{tRu}\} \wedge \mathsf{Pre}_a(\mathsf{u}) \leq \bigvee \{\blacklozenge^{\mathbb{A}} b \mid \mathsf{tRu}\} \wedge \mathsf{Pre}_a(\mathsf{u}) \\
&\leq \blacklozenge^{\mathbb{A}} b \wedge \mathsf{Pre}_a(\mathsf{u}) = \blacklozenge^{\mathbb{A}} b \wedge \mathsf{Pre}_a(\mathsf{u}) \wedge \mathsf{Pre}_a(\mathsf{u}) = (f_{\blacklozenge^{\mathbb{A}} b} \wedge \mathsf{Pre}_a)(\mathsf{u}) = [f_{\blacklozenge^{\mathbb{A}} b}](\mathsf{u}) \\
&= f^\varphi(\blacklozenge^{\mathbb{A}} b)(\mathsf{u}) = (f^\varphi \circ \blacklozenge^{\mathbb{A}} b)(\mathsf{u}).
\end{aligned}
$$

(ii) Let $[h], [k] \in \mathbb{A}^\varphi$, assume that $[h] \leq [k]$. Hence, $h(\mathsf{u}) \wedge \mathsf{Pre}_a(\mathsf{u}) \leq k(\mathsf{u}) \wedge \mathsf{Pre}_a(\mathsf{u})$ for every $\mathsf{u} \in \mathsf{S}$. Then, $\bigvee_{\mathsf{u} \in \mathsf{S}} (h(\mathsf{u}) \wedge \mathsf{Pre}_a(\mathsf{u})) \leq \bigvee_{\mathsf{u} \in \mathsf{S}} (k(\mathsf{u}) \wedge \mathsf{Pre}_a(\mathsf{u}))$, which proves that $g^\varphi([h]) \leq g^\varphi([k])$.

Let $[h_i] \in \mathbb{A}^\varphi$, where $i \in I$ for an index set I. Then we have that

$$
\begin{aligned}
&g^\varphi(\bigvee_{i \in I}[h_i]) = g^\varphi([\bigvee_{i \in I} h_i]) = \bigvee_{\mathsf{u} \in \mathsf{S}} \left(\bigvee_{i \in I} h_i(\mathsf{u}) \wedge \mathsf{Pre}_a(\mathsf{u})\right) \\
&= \bigvee_{\mathsf{u} \in \mathsf{S}} \left(\bigvee_{i \in I} (h_i(\mathsf{u}) \wedge \mathsf{Pre}_a(\mathsf{u})\right) = \bigvee_{i \in I} \left(\bigvee_{\mathsf{u} \in \mathsf{S}} (h_i(\mathsf{u}) \wedge \mathsf{Pre}_a(\mathsf{u})\right) = \bigvee_{i \in I} (g^\varphi(h_i)) \\
&\hfill (\mathsf{S} \text{ is finite})
\end{aligned}
$$

(iii) Let $[h] \in \mathbb{A}^\varphi$ and $b \in \mathbb{A}$, then we need to show that $[h] \leq f^\varphi(b)$ iff $g^\varphi([h]) \leq b$. By definition of f^φ, $[h] \leq f^\varphi(b)$ iff $[h] \leq [f_b]$. It follows from [21, Fact 9.2] that $[h] \leq [f_b]$ iff $h \wedge \mathsf{Pre}_a \leq f_b \wedge \mathsf{Pre}_a$. From this we obtain: for every $\mathsf{u} \in \mathsf{S}$,

$$
\begin{aligned}
h(\mathsf{u}) \wedge \mathsf{Pre}_a(\mathsf{u}) \leq f_b(\mathsf{u}) \wedge \mathsf{Pre}_a(\mathsf{u}) \quad &\text{iff} \quad h(\mathsf{u}) \wedge \mathsf{Pre}_a(\mathsf{u}) \leq (b \wedge \mathsf{Pre}_a(\mathsf{u})) \wedge \mathsf{Pre}_a(\mathsf{u}) \\
&\text{iff} \quad h(\mathsf{u}) \wedge \mathsf{Pre}_a(\mathsf{u}) \leq b \wedge \mathsf{Pre}_a(\mathsf{u}) \leq b \\
&\text{iff} \quad \bigvee_{\mathsf{u} \in \mathsf{S}} h(\mathsf{u}) \wedge \mathsf{Pre}_a(\mathsf{u}) \leq b \\
&\text{iff} \quad g^\varphi([h]) \leq b
\end{aligned}
$$

Proof of Lemma 5.4. Assume that $f : \mathbb{A} \to \mathbb{A}'$ is a refinement morphism. Define the relation $\mathfrak{R} = \{(s, s') \in S \times S' \mid s' \in f(\{s\})\}$. It is easy to see that $\mathfrak{R}$ is a refinement.

For the other direction, assume that $M_s \succeq M'_{s'}$. Hence the is a refinement relation $\mathfrak{R}$ from M_s to $M'_{s'}$. Define $f : \mathbb{A} \to \mathbb{A}'$ such that $f(X) = \mathfrak{R}[X]$, for every $X \subseteq S$. It is easy to see that f is a refinement morphism.

Proof of Lemma 5.7. (i) We show that F is a refinement morphism. Since for each $\varphi \in \mathcal{L}_{\Box\forall}$, f^{φ} is monotone, and preserves $\bot^{\mathbb{A}^{\varphi}}$ and $\vee^{\mathbb{A}^{\varphi}}$, it follows that $\prod_{\varphi\in\mathcal{L}_{\Box\forall}} f^{\varphi}$ also satisfies those conditions. It remains to prove that $\blacklozenge^{\mathfrak{A}} \circ F \leq F \circ \blacklozenge^{\mathbb{A}}$. Now note that $\blacklozenge^{\mathbb{A}^{\varphi}}(f^{\varphi}) \leq f^{\varphi}(\blacklozenge^{\mathbb{A}})$ for every $\varphi \in \mathcal{L}_{\Box\forall}$. Let $b \in \mathbb{A}$, then we have that

$$\blacklozenge^{\mathfrak{A}} \circ F(b) = \blacklozenge^{\mathfrak{A}}\big(f^{\varphi}(b)\big)_{\varphi\in\mathcal{L}_{\Box\forall}} = \big(\blacklozenge^{\mathbb{A}^{\varphi}}(f^{\varphi})(b)\big)_{\varphi\in\mathcal{L}_{\Box\forall}} \leq \\ \big(f^{\varphi}(\blacklozenge^{\mathbb{A}}(b))\big)_{\varphi\in\mathcal{L}_{\Box\forall}} = \prod_{\varphi\in\mathcal{L}_{\Box\forall}}\big(f^{\varphi}(\blacklozenge^{\mathbb{A}}(b))\big) = (\prod_{\varphi\in\mathcal{L}_{\Box\forall}} f^{\varphi})\big(\blacklozenge^{\mathbb{A}}(b)\big) = F \circ \blacklozenge^{\mathbb{A}}(b).$$

(ii) It is easy to see that G is monotone and preserves $\bot$. Also, G preserves $\top$, since the action model constructed for $\top$ is defined as follows: $\alpha^{\top} = (\mathsf{S}^{\top}, \mathsf{R}^{\top}, \mathsf{Pre}^{\top})$, where $\mathsf{S}^{\top} = \{skip\}$, $\mathsf{R}^{\top} = \{(skip, skip)\}$, $\mathsf{Pre}^{\top}(skip) = \top$. Then, we have $g^{\top}([\![\top]\!]_{\mathbb{A}^{\top}}) = [\![\mathsf{Pre}(skip)]\!]_{\mathbb{A}} = \top^{\mathbb{A}}$. We proceed to show that G preserves binary joins and then by induction we can easily prove that it preserves finite joins. Let $([h]^{\varphi})_{\varphi\in\mathcal{L}_{\Box\forall}}, ([k]^{\varphi})_{\varphi\in\mathcal{L}_{\Box\forall}} \in \mathfrak{A}$. Then

$$\begin{aligned} &G\Big((([h]^{\varphi})_{\varphi} \vee ([k]^{\varphi})_{\varphi}) \Big) = \bigvee_{\varphi\in\mathcal{L}_{\Box\forall}} g^{\varphi}\Big((([h]^{\varphi})_{\varphi} \vee ([k]^{\varphi})_{\varphi\in\mathcal{L}_{\Box\forall}}) \Big) \\ &= \bigvee_{\varphi\in\mathcal{L}_{\Box\forall}} g^{\varphi}([h]^{\varphi})_{\varphi} \vee g^{\varphi}([k]^{\varphi})_{\varphi} \\ &= \bigvee_{\varphi\in\mathcal{L}_{\Box\forall}} g^{\varphi}([h]^{\varphi})_{\varphi} \vee \bigvee_{\varphi\in\mathcal{L}_{\Box\forall}} g^{\varphi}([k]^{\varphi})_{\varphi} = G\big(([h]^{\varphi})_{\varphi} \vee G\big(([k]^{\varphi})_{\varphi}\big). \end{aligned}$$

(iii) $F \dashv G$ follows from the fact that $f^{\varphi} \dashv g^{\varphi}$, for each $\varphi \in \mathcal{L}_{\Box\forall}$.

Lemma A.1 *For any formula $\varphi \in \mathcal{L}_{\forall\Box}$, we have: $G([\![\Diamond\varphi]\!]'_{\mathcal{A}'}) \leq \Diamond G([\![\varphi]\!]'_{\mathcal{A}'})$.*

Proof. Fix an algebraic model $\mathcal{A}$ and a formula $\varphi \in \mathcal{L}_{\forall\Box}$. We want to prove $G([\![\Diamond\varphi]\!]'_{\mathcal{A}'}) \leq \Diamond G([\![\varphi]\!]'_{\mathcal{A}'})$.

$$\begin{aligned} G([\![\Diamond\varphi]\!]'_{\mathcal{A}'}) = \bigvee_{\gamma} g^{\gamma}([\![\Diamond\varphi]\!]'_{\mathcal{A}^{\gamma}}) = \bigvee_{\gamma} \big([\![\langle\alpha^{\gamma}_{\mathsf{S}^{\gamma}}\rangle\Diamond\varphi]\!]'_{\mathcal{A}}\big) \quad &\text{(definition of } G \text{ and } g^{\gamma}) \\ = \bigvee_{\gamma}\bigvee_{\mathsf{u}\in\mathsf{S}^{\gamma}} \left([\![\mathsf{Pre}^{\gamma}(\mathsf{u})]\!]'_{\mathcal{A}} \wedge [\![\bigvee_{\mathsf{v}\in\mathsf{R}^{\gamma}(\mathsf{u})} \Diamond\langle\alpha^{\gamma}_{\mathsf{v}}\rangle\varphi]\!]'_{\mathcal{A}} \right) \quad &\text{(A.1)} \end{aligned}$$

$$\leq \bigvee_{\gamma} \bigvee_{\mathsf{u}\in\mathsf{S}^\gamma} [\![\bigvee_{\mathsf{v}\in\mathsf{R}^\gamma(\mathsf{u})} \Diamond\langle\alpha^\gamma_{\mathsf{v}}\rangle\varphi]\!]'_{\mathcal{A}} \qquad (a \wedge b \leq a)$$

$$\leq \bigvee_{\gamma} \bigvee_{\mathsf{u}\in\mathsf{S}^\gamma} \Diamond[\![\bigvee_{\mathsf{v}\in\mathsf{R}^\gamma(\mathsf{u})} \langle\alpha^\gamma_{\mathsf{v}}\rangle\varphi]\!]'_{\mathcal{A}} \qquad (\Diamond(\varphi \vee \psi) = \Diamond\varphi \vee \Diamond\psi)$$

$$\leq \bigvee_{\gamma} \bigvee_{\mathsf{u}\in\mathsf{S}^\gamma} \Diamond[\![\bigvee_{\mathsf{u}\in\mathsf{S}^\gamma} \langle\alpha^\gamma_{\mathsf{u}}\rangle\varphi]\!]'_{\mathcal{A}} \qquad (\mathsf{R}^\gamma(\mathsf{u}) \subseteq \mathsf{S}^\gamma)$$

$$= \bigvee_{\gamma} \bigvee_{\mathsf{u}\in\mathsf{S}^\gamma} \Diamond g^\gamma([\![\varphi]\!]'_{\mathcal{A}^\gamma}) \qquad (\text{definition of } g^\gamma)$$

$$= \bigvee_{\gamma} \Diamond g^\gamma([\![\varphi]\!]'_{\mathcal{A}^\gamma}) = \Diamond \bigvee_{\gamma} g^\gamma([\![\varphi]\!]'_{\mathcal{A}^\gamma})$$

$$= \Diamond G([\![\varphi]\!]'_{\mathcal{A}'}) \qquad (\text{definition of } G)$$

□

Equivalence (A.1) follows from $\langle\alpha_u\rangle\varphi \leftrightarrow \mathsf{Pre}(\mathsf{u}) \wedge \bigvee_{\mathsf{v}\in\mathsf{R}(\mathsf{u})}\langle\alpha_v\rangle\varphi$ [21, Page 14].

Proof of Theorem 5.9.
Soundness. The definition of $[\![\cdot]\!]'_{\mathcal{A}}$ for $\mathcal{L}_\Box$ is identical to the algebraic semantics proposed in [21]. Hence the axioms and rules of K are sound w.r.t. this semantics.

(i) Axiom **RProp**. We need to prove $[\![\exists p]\!]'_{\mathcal{A}} = [\![p]\!]'_{\mathcal{A}}$ and $[\![\exists\neg p]\!]'_{\mathcal{A}} = [\![\neg p]\!]'_{\mathcal{A}}$. For every $p \in \mathsf{AtProp}$, $g^\varphi([\![p]\!]'_{\mathcal{A}^\varphi}) \leq [\![p]\!]'_{\mathcal{A}}$, so that $\bigvee_{\varphi\in\mathcal{L}_{\Box\forall}} g^\varphi([\![p]\!]'_{\mathcal{A}^\varphi}) \leq [\![p]\!]'_{\mathcal{A}}$. So, $G([\![p]\!]'_{\mathcal{A}'}) \leq [\![p]\!]'_{\mathcal{A}}$. For the other direction, according to the construction of multi-pointed action model $\alpha^p_{\mathsf{S}^p}$ for the atomic proposition p [18, Lemma V.2], $[\![\langle\alpha^p_{\mathsf{S}^p}\rangle p]\!]'_{\mathcal{A}} = [\![p]\!]'_{\mathcal{A}}$. This implies that $[\![p]\!]'_{\mathcal{A}} = g^p([\![p]\!]'_{\mathcal{A}^p})$ and $[\![p]\!]'_{\mathcal{A}} \leq \bigvee_{\varphi\in\mathcal{L}_{\Box\forall}} g^\varphi([\![p]\!]'_{\mathcal{A}^\varphi}) = G([\![p]\!]'_{\mathcal{A}'})$ as required. The other equality can be proved in a similar way.

(ii) Axiom **R**. We need to show $[\![\forall(\varphi \to \psi)]\!]' \leq [\![\forall\varphi \to \forall\psi]\!]'$.
First, observe that $[\![\forall(\varphi \to \psi)]\!]'_{\mathcal{A}} = [\![\neg\exists\neg(\varphi \to \psi)]\!]'_{\mathcal{A}} = \neg G([\![\neg(\varphi \to \psi)]\!]'_{\mathcal{A}'})$ and $[\![\forall\varphi \to \forall\psi]\!]'_{\mathcal{A}} = [\![\exists\neg\varphi \vee \neg\exists\neg\psi]\!]'_{\mathcal{A}} = G([\![\neg\varphi]\!]'_{\mathcal{A}'}) \vee \neg G([\![\neg\psi]\!]'_{\mathcal{A}'})$.
Hence, it is enough to show that

$$\neg G([\![\neg(\varphi \to \psi)]\!]'_{\mathcal{A}'}) \leq G([\![\neg\varphi]\!]'_{\mathcal{A}'}) \vee \neg G([\![\neg\psi]\!]'_{\mathcal{A}'})$$

First note that $\neg\varphi\vee\neg\psi \leftrightarrow \neg\varphi\vee\neg(\varphi \to \psi)$. So, $[\![\neg\varphi\vee\neg\psi]\!]'_{\mathcal{A}'} = [\![\neg\varphi\vee\neg(\varphi \to \psi]\!]'_{\mathcal{A}'}$. Then, $G([\![\neg\varphi\vee\neg\psi]\!]'_{\mathcal{A}'}) = G([\![\neg\varphi\vee\neg(\varphi \to \psi)]\!]'_{\mathcal{A}'}))$. Since G preserves $\vee$, we get

$$G([\![\neg\varphi]\!]'_{\mathcal{A}'}) \vee G([\![\neg\psi]\!]'_{\mathcal{A}'}) = G([\![\neg\varphi]\!]'_{\mathcal{A}'}) \vee G([\![\neg(\varphi \to \psi)]\!]'_{\mathcal{A}'})).$$

By applying negation and DeMorgan laws, we get

$$\neg G([\![\neg\varphi]\!]'_{\mathcal{A}'}) \wedge \neg G([\![\neg\psi]\!]'_{\mathcal{A}'}) = \neg G([\![\neg\varphi]\!]'_{\mathcal{A}'}) \wedge \neg G([\![\neg(\varphi \to \psi)]\!]'_{\mathcal{A}'})$$

The equality above implies

$$\neg G([\![\neg\varphi]\!]'_{\mathcal{A}'}) \wedge \neg G([\![\neg(\varphi \to \psi)]\!]'_{\mathcal{A}'}) \leq \neg G([\![\neg\varphi]\!]'_{\mathcal{A}'}) \wedge \neg G([\![\neg\psi]\!]'_{\mathcal{A}'}).$$

It is easy to see that in any **BA**, $a\wedge b \leq a\wedge c$ implies that $b \leq \neg a\vee c$, which implies $\neg G([\![\neg(\varphi \to \psi)]\!]'_{\mathcal{A}'}) \leq G([\![\neg\varphi]\!]'_{\mathcal{A}'}) \vee \neg G([\![\neg\psi]\!]'_{\mathcal{A}'})$, as required.

(iii) Axiom **RK**. We need to show that $\llbracket\exists\nabla\Phi\rrbracket'_{\mathcal{A}} = \llbracket\bigwedge \Diamond\exists\Phi\rrbracket'_{\mathcal{A}}$ for every algebraic model $\mathcal{A}$. Fix an algebraic model $\mathcal{A}$.

Proof of $\llbracket\exists\nabla\Phi\rrbracket'_{\mathcal{A}} \leq \llbracket\bigwedge \Diamond\exists\Phi\rrbracket'_{\mathcal{A}}$.

$$\begin{aligned}
\llbracket\exists\nabla\Phi\rrbracket'_{\mathcal{A}} &= G\left(\llbracket\Box\left(\bigvee_{\varphi\in\Phi}\varphi\right) \wedge \bigwedge_{\varphi\in\Phi}\Diamond\varphi\rrbracket'_{\mathcal{A}'}\right) && (\text{with } G : \mathfrak{A}_{\mathcal{A}} \to \mathbb{A})\\
&\leq G\left(\llbracket\bigwedge_{\varphi\in\Phi}\Diamond\varphi\rrbracket'_{\mathcal{A}'}\right) \leq \bigwedge_{\varphi\in\Phi} G(\llbracket\Diamond\varphi\rrbracket'_{\mathcal{A}'}) && (\text{monotonicity of } G)\\
&\leq \bigwedge_{\varphi\in\Phi}\Diamond G(\llbracket\varphi\rrbracket'_{\mathcal{A}'}) = \llbracket\bigwedge\Diamond\exists\Phi\rrbracket'_{\mathcal{A}}. && (\text{Lemma A.1})
\end{aligned}$$

Proof of $\llbracket\bigwedge \Diamond\exists\Phi\rrbracket'_{\mathcal{A}} \leq \llbracket\exists\nabla\Phi\rrbracket'_{\mathcal{A}}$.

Let Φ be a finite set of formulas. We first show that $\bigwedge\Diamond G(\llbracket\Phi\rrbracket'_{\mathcal{A}'}) \leq g^{\nabla\Phi}(\llbracket\nabla\Phi\rrbracket'_{\mathcal{A}^{\nabla\Phi}})$. In order to show this we use the inductive structure of formulas $\nabla\Phi$ that may contain action models [18, Lemma V.2] and the algebraic semantics of AML. Let $\Phi \subseteq \mathcal{L}_{\Box\forall}$ be a set of formulas, and for each $\varphi \in \Phi$, $\alpha^{\varphi} = (\mathsf{S}^{\varphi}, \mathsf{R}^{\varphi}, \mathsf{Pre}^{\varphi})$, $\alpha^{\nabla\Phi} = (\mathsf{S}^{\nabla\Phi}, \mathsf{R}^{\nabla\Phi}, \mathsf{Pre}^{\nabla\Phi})$ be action models for the formulas $\varphi \in \Phi$ and $\nabla\Phi$, respectively. Note that $\mathsf{S}^{\nabla\Phi} = \{\mathsf{u}^{\star}\} \cup \bigcup_{\varphi\in\Phi}\mathsf{S}^{\varphi}$, $\mathsf{R}^{\nabla\Phi} = \{(\mathsf{u}^{\star}, \mathsf{u}) \mid \varphi \in \Phi, \mathsf{u} \in \mathsf{S}^{\varphi}\} \cup \bigcup \mathsf{R}^{\varphi}$ and $\mathsf{Pre}^{\nabla\Phi} = \{(\mathsf{u}^{\star}, \Diamond\exists\varphi \wedge \Diamond\exists\psi)\} \cup \bigcup \mathsf{Pre}^{\varphi}$. Then,

$$\begin{aligned}
g^{\nabla\Phi}(\llbracket\nabla\Phi\rrbracket'_{\mathcal{A}^{\nabla\Phi}}) &= \bigvee_{\mathsf{u}\in\mathsf{S}^{\nabla\Phi}} \llbracket\langle\alpha^{\nabla\Phi}\rangle\nabla\Phi\rrbracket'_{\mathcal{A}}\\
&= \llbracket\langle\alpha^{\nabla\Phi}_{\mathsf{u}^{\star}}\rangle\nabla\Phi\rrbracket'_{\mathcal{A}} \vee \bigvee_{\varphi\in\Phi}\bigvee_{\mathsf{u}\in\mathsf{S}^{\varphi}} \llbracket\langle\alpha^{\nabla\Phi}_{\mathsf{u}}\rangle\nabla\Phi\rrbracket'_{\mathcal{A}}.
\end{aligned}$$

Also,

$$\begin{aligned}
\llbracket\langle\alpha^{\nabla\Phi}_{\mathsf{u}^{\star}}\rangle\nabla\Phi\rrbracket'_{\mathcal{A}} &= \llbracket\langle\alpha^{\nabla\Phi}_{\mathsf{u}^{\star}}\rangle(\Box(\bigvee\Phi) \wedge \bigwedge\Diamond\Phi)\rrbracket'_{\mathcal{A}}\\
&= \llbracket\langle\alpha^{\nabla\Phi}_{\mathsf{u}^{\star}}\rangle\Box(\bigvee\Phi)\rrbracket'_{\mathcal{A}} \wedge \llbracket\langle\alpha^{\nabla\Phi}_{\mathsf{u}^{\star}}\rangle\bigwedge\Diamond\Phi\rrbracket'_{\mathcal{A}}
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\llbracket\langle\alpha^{\nabla\Phi}_{\mathsf{u}^{\star}}\rangle\Box(\bigvee\Phi)\rrbracket'_{\mathcal{A}} = \llbracket\mathsf{Pre}^{\nabla\Phi}(\mathsf{u}^{\star})\rrbracket'_{\mathcal{A}} \wedge \llbracket\bigwedge_{\mathsf{u}\in\mathsf{R}^{\nabla\Phi}(\mathsf{u}^{\star})}\Box[\alpha^{\nabla\Phi}_{\mathsf{u}}](\bigvee\Phi)\rrbracket'_{\mathcal{A}}\\
&= \llbracket\mathsf{Pre}^{\nabla\Phi}(\mathsf{u}^{\star})\rrbracket'_{\mathcal{A}} \wedge \Box^{\mathbb{A}}\bigwedge_{\mathsf{u}\in\mathsf{R}^{\nabla\Phi}(\mathsf{u}^{\star})}\llbracket[\alpha^{\nabla\Phi}_{\mathsf{u}}](\bigvee\Phi)\rrbracket'_{\mathcal{A}}\\
&= \llbracket\mathsf{Pre}^{\nabla\Phi}(\mathsf{u}^{\star})\rrbracket'_{\mathcal{A}} \wedge \Box^{\mathbb{A}}\left(\bigwedge_{\varphi\in\Phi}\bigwedge_{\mathsf{u}\in\mathsf{S}^{\varphi}}\llbracket[\alpha^{\nabla\Phi}_{\mathsf{u}}](\bigvee\Phi)\rrbracket'_{\mathcal{A}}\right)\\
&= \llbracket\mathsf{Pre}^{\nabla\Phi}(\mathsf{u}^{\star})\rrbracket'_{\mathcal{A}} \wedge \Box^{\mathbb{A}}\left(\bigwedge_{\varphi\in\Phi}\bigwedge_{\mathsf{u}\in\mathsf{S}^{\varphi}}\llbracket[\alpha^{\varphi}_{\mathsf{u}}](\bigvee\Phi)\rrbracket'_{\mathcal{A}}\right)\\
&= \llbracket\mathsf{Pre}^{\nabla\Phi}(\mathsf{u}^{\star})\rrbracket'_{\mathcal{A}} \wedge \Box^{\mathbb{A}}\left(\bigwedge_{\varphi\in\Phi}\llbracket[\alpha^{\varphi}_{\mathsf{S}^{\varphi}}](\bigvee\Phi)\rrbracket'_{\mathcal{A}}\right)
\end{aligned}$$

It follows from the definition of the structure of action models $\alpha^{\nabla\Phi}$ and α^{φ} that $\vdash [\alpha^{\varphi}]\varphi$, for every $\varphi \in \Phi$. So, we have $\vdash [\alpha^{\varphi}](\bigvee \Phi)$ which means that $[\![[\alpha^{\varphi}](\bigvee \Phi)]\!]'_{\mathcal{A}} = \top$. Therefore,

$$[\![\langle \alpha_{\mathsf{u}^\star}^{\nabla\Phi}\rangle \Box(\bigvee \Phi)]\!]'_{\mathcal{A}} = [\![\mathsf{Pre}^{\nabla\Phi}(\mathsf{u}^\star)]\!]'_{\mathcal{A}} = [\![\bigwedge \Diamond\exists\Phi]\!]'_{\mathcal{A}}. \tag{A.2}$$

For every pointed action model $\alpha_{\mathsf{u}} = (\mathsf{S}, \mathsf{R}, \mathsf{Pre})$ over $\mathbb{A}$,

$$[\![\langle\alpha_{\mathsf{u}}\rangle\gamma]\!]'_{\mathcal{A}} = [\![\neg[\alpha_{\mathsf{u}}]\neg\gamma]\!]'_{\mathcal{A}} = [\![\neg\,(\mathsf{Pre}(\mathsf{u}) \to \neg[\alpha_{\mathsf{u}}]\gamma)]\!]'_{\mathcal{A}} = [\![\mathsf{Pre}(\mathsf{u}) \wedge [\alpha_{\mathsf{u}}]\gamma]\!]'_{\mathcal{A}}$$

So we have $[\![\langle \alpha_{\mathsf{u}^\star}^{\nabla\Phi}\rangle \Diamond\varphi]\!]'_{\mathcal{A}} = [\![\mathsf{Pre}^{\nabla\Phi}(\mathsf{u}^\star)]\!]'_{\mathcal{A}} \wedge [\![[\alpha_{\mathsf{u}^\star}^{\nabla\Phi}]\Diamond\varphi]\!]'_{\mathcal{A}}$ Since $\vdash \bigwedge[\alpha_{\mathsf{u}^\star}^{\nabla\Phi}]\Diamond\Phi$ [18, proof of Lemma V.2], we can deduce that for every $\varphi \in \Phi$, $\vdash [\alpha_{\mathsf{u}^\star}^{\nabla\Phi}]\Diamond\varphi$. We then get that for every $\varphi \in \Phi$,

$$[\![\langle \alpha_{\mathsf{u}^\star}^{\nabla\Phi}\rangle \Diamond\varphi]\!]'_{\mathcal{A}} = [\![\mathsf{Pre}^{\nabla\Phi}(\mathsf{u}^\star)]\!]'_{\mathcal{A}}$$

which together with (A.2) yields that

$$[\![\langle \alpha_{\mathsf{u}^\star}^{\nabla\Phi}\rangle \nabla\Phi]\!]_{\mathcal{A}} = [\![\mathsf{Pre}^{\nabla\Phi}(\mathsf{u}^\star)]\!]'_{\mathcal{A}} = [\![\Diamond\exists\varphi]\!]'_{\mathcal{A}} \wedge [\![\Diamond\exists\psi]\!]'_{\mathcal{A}}$$

To complete the proof, we have

$$\begin{aligned} G([\![\nabla\Phi]\!]'_{\mathcal{A}'}) &= \bigvee_{\gamma \in \mathcal{L}_{\forall\Box}} g^{\gamma}([\![\nabla\Phi]\!]'_{\mathcal{A}^{\gamma}}) \geq g^{\nabla\Phi}([\![\nabla\Phi]\!]'_{\mathcal{A}^{\nabla\Phi}}) \\ &\geq [\![\langle \alpha_{\mathsf{u}^\star}^{\nabla\Phi}\rangle \nabla\Phi]\!]'_{\mathcal{A}} = \bigwedge [\![\Diamond\exists\Phi]\!]'_{\mathcal{A}} = \bigwedge \Diamond G([\![\Phi]\!]'_{\mathcal{A}'}) \end{aligned}$$

Completeness. RML contains all the axioms of K. The algebraic semantics is complete w.r.t. K. RML is equivalent to K, hence the algebraic semantics is complete w.r.t. RML.

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