

# Planarity of Additively Drawn Concept Lattices

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**Abstract.** In Formal Concept Analysis, it is a very important but quite difficult task to draw line diagrams of concept lattices automatically. In particular, we want every planar lattice to be visualized without edge crossings. Many algorithms ignore that fact or find plane diagrams only heuristically. We present a characterization of planar lattices based on the theorem of Baker, Fishburn and Roberts [1] and the "left"-relation introduced by Rival [5]. In particular, our work is helpful for drawing attribute- (or dually object-) additive diagrams.

## 1 Motivation

There exist algorithms for embedding lattices into the plane already, see [2] for an overview. These algorithms do not use the lattice structure, but more general ones (graphs or acyclic digraphs) for the characterisation of planar lattices. We want to use the convention of attribute-additively drawn diagrams [3], because it provides nice visualizations of lattices. By applying this rule, the coordinates of the nodes of the attribute concepts (denoted by *attribute vector* in the following) define the coordinates of all other nodes. Hence the relationship of the attribute vectors in the plane appears to characterize the planarity of the diagram.

## 2 Introduction

In the following we always deal with finite lattices. As we are interested in algorithms for visualizing lattices, this seems to be sufficient. When speaking about contexts, we always mean reduced ones. By a diagram of a lattice  $\mathfrak{L}$  we actually mean an upward line diagram, which is denoted by  $\text{pos}(\mathfrak{L})$ .

One base of our work is the theorem of Baker, Fishburn and Roberts [1]:

**Theorem 1** *A lattice is planar if and only if it has an order dimension of at most two.*

This theorem allows us to characterize planar lattices by *interval-inclusion-lattices (IIL)*.

**Definition 2** *A lattice  $\mathfrak{L}$  is an IIL, if  $\mathfrak{L} \cong (I, \subseteq)$  holds for a subset  $I \subseteq \mathcal{I}(\mathbb{R})$ . The set  $\mathcal{I}(\mathbb{R})$  denotes the set of all closed intervals in  $\mathbb{R}$ .*

**Corollary 3** [4] *A finite lattice is planar if and only if it is isomorphic to an IIL.*

When drawing lattices, most people tend to use (at least partially) the convention of attribute (or dually object) additivity, even if they do not know what it actually is. This method is useful mainly for distributive (or "nearly distributive") lattices, as the resulting diagrams look like drawn on an  $n$ -dimensional grid [6].

**Definition 4** *Let  $\mathfrak{B}(G, M, I)$  be a concept lattice. A diagram  $\text{pos}(\mathfrak{B}(G, M, I))$  is attribute additive, if there is a map  $\text{vec} : M \mapsto \mathbb{R}^2$ , such that the equation*

$$\text{pos}(\mathfrak{c}) = \sum_{m \in \text{Int}(\mathfrak{c})} \text{vec}(m)$$

*holds for all concepts  $\mathfrak{c} \in \mathfrak{B}(G, M, I)$ .*

### 3 The $L$ -Relation

In the following, we introduce binary relations  $L(\mathfrak{A})$  and  $L(\text{pos}(\mathfrak{A}))$  ( $L$  is an abbreviation for "left") on elements of lattices  $\mathfrak{A}$  and nodes of line diagrams  $\text{pos}(\mathfrak{A})$ . Dually, we can consider a relation  $R$  (which stands for "right") and naturally assume, that  $eLf \iff fRe$  holds for  $e, f$  being arbitrary lattice elements or diagram nodes respectively.

#### 3.1 The $L$ -Relation on Lattices

When having a closer look on IILs, we can easily give a definition for an interval being left of another one.

**Definition 5** *Let  $\mathcal{I}$  be an IIL. Two elements  $[i_l, i_r], [j_l, j_r] \in \mathcal{I}$  are in  $L$ -relation, if the conditions  $i_l < j_l$  and  $i_r < j_r$  hold.*

Obviously, this relation is a strict order. We also notice, that two intervals are in  $L$ -relation, if and only if they are incomparable. Indeed we can proof that any finite lattice possessing a relation, which fulfills these requirements is planar already.

**Corollary 6** *A lattice  $\mathfrak{A} = (\mathfrak{A}, \leq)$  is planar, if and only if there is a binary relation  $L$  on  $\mathfrak{A}$ , such that the below-mentioned conditions hold:*

$$L \text{ is a strict order.} \tag{1}$$

$$L \cup L^{-1} = \parallel \tag{2}$$

*Proof.* " $\Rightarrow$ ": This can be derived immediately from Theorem 3 and Definition 5. " $\Leftarrow$ ": Consider the relations  $L_{<} := L \cup <$  and  $R_{<} := R \cup <$ . We show the following properties:

1.  $L_{<}$  is connex because of condition (2).
2.  $L_{<}$  is irreflexive, since  $L$  and  $<$  are irreflexive.
3. From  $vL_{<}w$  and  $wL_{<}v$  follows neither  $vLw$  and  $wLv$  nor  $v < w$  and  $w < v$ , since  $L$  and  $<$  are orders. The remaining cases infringe condition (2). Hence  $L_{<}$  is antisymmetric.
4. Assume  $vL_{<}wL_{<}zL_{<}v$ . Since  $L$  and  $<$  are strict orders, we conclude w.l.o.g.  $vLw < z$ . This is contrary to condition 2, since we have either  $z < v$ , i.e.  $w < v$  or  $zLv$ , i.e.  $zLw$ . Hence  $L_{<}$  is transitive.

We conclude that  $L_{<}$  is a linear strict order. By applying analogous argumentation, we realize that  $R_{<}$  is a linear strict order too. Additionally we see the identity  $L_{<} \cap R_{<} = <$ . Hence the order dimension of  $\underline{\mathfrak{B}}$  is at most two and with Theorem 1 we conclude that  $\underline{\mathfrak{B}}$  is planar.

How can we find  $L$ -relations fulfilling the conditions from corollary 6? We come back to the already mentioned attributive additive approach. First we describe that relation only for attribute concepts with common upper neighbour. Here the effect of a node  $v$  being left of another node  $w$  is intuitively understandable: in an appropriate diagram the line connecting  $v$  with its upper neighbour  $v^*$  should be left of the line connecting  $w$  and  $v^*$ . However, now we are interested only in lattices and not yet in diagrams, so the following definitions will be more abstract.

**Definition 7** Let  $\underline{\mathfrak{B}}(G, M, I)$  be a concept lattice. A relation  $L_a \subseteq M \times M$  is called sorting relation, if the following condition holds for all attributes  $m_i, m_j \in M$ :

$$\mu m_i^* = \mu m_j^* \iff \text{either } m_i L_a m_j \text{ or } m_j L_a m_i \text{ holds.}$$

Now we extend the sorting relation to a bigger domain.

**Definition 8** Let  $\underline{\mathfrak{B}}(G, M, I)$  be a concept lattice with given sorting relation  $L_a$ . For arbitrary lattice elements  $v$  and  $w$ , we define  $M(v, w) = (Int(v) \setminus Int(w)) \times (Int(w) \setminus Int(v))$ . We define the relation  $L \subseteq \underline{\mathfrak{B}}(G, M, I) \times \underline{\mathfrak{B}}(G, M, I)$  as follows:

1.  $m_i L_a m_j \implies \mu m_i L \mu m_j$
2. If for attribute concepts,  $\mu m_i^* \neq \mu m_j^*$  holds,

$$\mu m_i L \mu m_j : \iff \exists (m_k, m_l) \in M(m_i, m_j) \setminus \{(m_i, m_j)\} : \mu m_k L \mu m_l.$$

3. After  $L$  is computed on all pairs of attribute concepts,

$$vLw : \iff \exists (m_i, m_j) \in M(v, w) : \mu m_i L \mu m_j.$$

Without proof, we give as a first result, that the above defined relation fulfills condition (2) of corollary 6:

**Lemma 9** For the  $L$ -relation from definition 8, the identity  $L \cup L^{-1} = \parallel$  holds.

On the other hand we can also proof, that the  $L$ -relation from definition 8 is (for a fixed sorting relation) the only one meeting the requirements of corollary 6.

**Lemma 10** *Let  $\mathfrak{B}(G, M, I)$  be a concept lattice and  $L \subseteq \mathfrak{B}(G, M, I) \times \mathfrak{B}(G, M, I)$  fulfilling the conditions (1) and (2). Then the following implication holds for all lattice elements  $v_1, v_2, w_1, w_2$ :*

$$v_1 L w_1 \wedge (v_1, w_1) \in M(v_2, w_2) \implies v_2 L w_2$$

*Proof.* We assume  $w_2 L v_2$ . Since  $(v_1, w_1) \in M(v_2, w_2)$  we know, that  $v_1$  and  $w_2$  are incomparable. We conclude either  $v_1 L w_2$  and  $v_1 L v_2$ , since  $L$  is transitive, or  $w_2 L v_1$  and  $v_2 L v_1$ . Both cases lead to a contradiction, since  $v_1$  and  $v_2$  are comparable.

The lemmata 9 and 10 point out, that the relations  $L$  are the only ones which may lead to a planar lattice. This gives us an (albeit very slow) algorithm to check whether a lattice is planar: Compute the  $L$ -relations from all possible sorting relations (at most  $|M|!$ ) and check, whether they are strict orders.

### 3.2 The $L$ -Relation on Diagrams

In the previous subsection we gave an idea how to define an additional relation  $L$  in a lattice to check, whether it is planar. Now we want to give a method to actually embed a planar lattice into the plane, if the relation  $L$  is given. First we have to define how this relation is to be found in a diagram.

**Definition 11** *Let  $\mathfrak{B}(G, M, I)$  be a concept lattice with a diagram  $\text{pos}(\mathfrak{B}(G, M, I))$ . The sorting relation  $L_a(\text{pos}(\mathfrak{B}(G, M, I))) \subseteq \text{pos}(M) \times \text{pos}(M)$  is defined as follows ( $\varphi(e)$  denotes the angle of the line  $e$ ):*

$$\begin{aligned} \text{pos}(m_i) L_a \text{pos}(m_j) : &\iff \\ &m_i^* = m_j^* \wedge \varphi(\text{pos}(m_i) - \text{pos}(m_i^*)) < \varphi(\text{pos}(m_j) - \text{pos}(m_j^*)). \end{aligned}$$

This definition is indeed very similar to definition 7. Every sorting relation in a diagram defines a sorting relation on the underlying lattice, and every sorting relation in a lattice can be realized in a diagram. Now we again extend this relation to an  $L$ -relation such that  $L \cup L^{-1} = \parallel$ :

**Definition 12** *Let  $\mathfrak{B}(G, M, I)$  be a concept lattice with a diagram  $\text{pos}(\mathfrak{B}(G, M, I))$ . Let  $p = 0_{\mathfrak{B}(G, M, I)} \dots 1_{\mathfrak{B}(G, M, I)}$  be an arbitrary chain from bottom to top. We denote with  $F_l(p)$  the area left of  $p$  and with  $F_r(p)$  the area right of  $p$ . We define the binary relation  $L$ , such that*

$$v L w : \iff (\exists p \ni w : v \in F_l(p)) \wedge (v \parallel w)$$

*holds for all diagram nodes  $v$  and  $w$ .*

This definition is quite similar to the one I. Rival gave in [5]<sup>1</sup>. In figure 1 we provide two examples for the  $L$ -relation of a diagram.

After these preparations we come to the main result of that work. It gives a possibility to actually draw a plane diagram of a planar lattice, where the relation  $L$  is conserved.

<sup>1</sup> The differences are due to make this definition compatible to the one of  $L$ -relations in lattices.

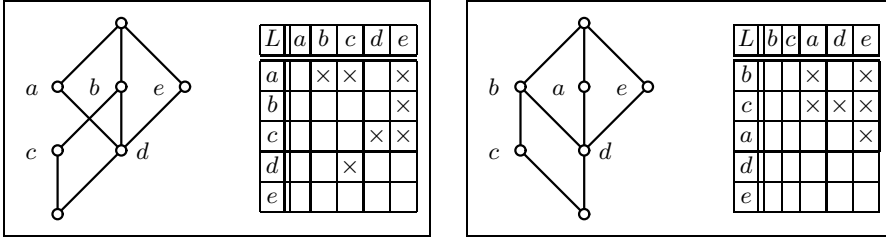


Fig. 1. Two diagrams of the same lattice with their  $L$ -relations.

**Theorem 13** Let  $\mathfrak{B}(G, M, I)$  be a concept lattice,  $L_a$  a sorting relation in  $\mathfrak{B}(G, M, I)$  and  $L$  the associated  $L$ -relation. The following statements are equivalent:

1. There exists a plane diagram with the sorting relation  
 $L_a(\text{pos}(\mathfrak{B}(G, M, I))) = \text{pos}(L_a(\mathfrak{B}(G, M, I)))$
2.  $L$  is a strict order.

*Proof.* " $\Leftarrow$ ": By using the relations  $L_{<}$  and  $R_{<}$  from the proof of corollary 6 we can define a map  $\psi : \mathfrak{B}(G, M, I) \mapsto \mathcal{I}$ , where  $\mathcal{I}$  is a IIL, by  $\psi : v \mapsto [\psi_l(v), \psi_r(v)]$ , where the mappings  $\psi_l, \psi_r : \mathfrak{B}(G, M, I) \mapsto \mathbb{R}$  meet the conditions

$$vL_{<}w \implies \psi_r(v) < \psi_r(w) \text{ and } vR_{<}w \implies \psi_l(w) < \psi_l(v).$$

We can easily show, that  $\psi$  is an isomorphism between  $(\mathfrak{B}(G, M, I), \leq, L)$  and  $(\mathcal{I}, \subseteq, \tilde{L})$ , where  $\tilde{L}$  is the above-defined "left"-relation for IILs. We define another map  $\text{pos} : \psi(\mathfrak{B}(G, M, I)) \mapsto \mathbb{R}^2$  by  $\text{pos}([x, y]) := (x, y)$ . Now we can prove, that the diagram  $\text{pos}(\psi(\mathfrak{B}(G, M, I)))$  is plane and that its sorting relation is the restriction of  $L$ .

1.  $\text{pos}(\psi(\mathfrak{B}(G, M, I)))$  does not contain an edge crossing: We assume, that the diagram edges corresponding to the elements  $v_1 \prec v_3$  and  $v_2 \prec v_4$  cross. Let  $(x_i, y_i)$  the coordinates of the node  $v_i$  and  $(x_5, y_5)$  be the coordinates of the intersection. With the definition of  $\psi$  we conclude  $x_3, x_4 < x_5 < x_1, x_2$  and  $y_1, y_2 < y_5 < y_3, y_4$ , i.e.  $v_2 < v_3$  and  $v_1 < v_4$ . That means, that  $v_3$  and  $v_4$  do not have an infimum in contradiction to  $\mathfrak{B}(G, M, I)$  being a lattice.
2.  $\text{pos}(\psi(\mathfrak{B}(G, M, I)))$  is an upward drawing: this is obviously satisfied by the definitions of  $\text{pos}$  and  $\psi$ .
3. Let  $m_i L_a m_j$  for two attributes in  $M$ . Let  $(x_i, y_i)$  and  $(x_j, y_j)$  be the coordinates of the appropriate diagram nodes and  $(x, y)$  the coordinates of their common upper neighbour. Then the inequalities  $x < x_i < x_j$  and  $y_1 < y_2 < y$  hold. For the angles  $\varphi_i := \varphi(\text{pos}(\mu m_i) - \text{pos}(\mu m_i^*))$  and  $\varphi_j := \varphi(\text{pos}(\mu m_j) - \text{pos}(\mu m_j^*))$  we get

$$\tan \varphi_i = \frac{y_i - y}{x_i - x} \text{ and } \tan \varphi_j = \frac{y_j - y}{x_j - x}.$$

it arises  $y_i - y < y_j - y < 0$  and  $0 < x_i < x_j$ , i.e.  $\tan \varphi_i < \tan \varphi_j$ . The angles are in the interval  $(3/2 \cdot \pi, \pi)$ . In this domain the function  $\arctan$  is monotonous, so we conclude  $\varphi_i < \varphi_j$ , i.e.  $\text{pos}(m_i)L_a\text{pos}(m_j)$ .

" $\Rightarrow$ ":

1.  $L(\text{pos}(\underline{\mathfrak{B}}(G, M, I)))$  is irreflexive by definition.
2. We assume  $z_1Lz_2$  and  $z_2Lz_1$  for two diagram nodes  $z_1, z_2$ . Then we find two chains  $p_1 = \text{pos}(0_{\underline{\mathfrak{B}}(G, M, I)}) \dots z_1 \dots \text{pos}(1_{\underline{\mathfrak{B}}(G, M, I)})$  and  $p_2 = \text{pos}(0_{\underline{\mathfrak{B}}(G, M, I)}) \dots z_2 \dots \text{pos}(1_{\underline{\mathfrak{B}}(G, M, I)})$  such that  $p_1$  and  $p_2$  intersect. Since the diagram is plane, the point of intersection is again a node in contradiction of the concepts corresponding to  $z_1$  and  $z_2$  not being comparable. Hence  $L$  is antisymmetric.
3. In an analogous way we can show  $L$  to be transitive.
4. Every two incomparable nodes are in  $L$ -relation: this is obviously granted by definition.
5. If  $\text{pos}(m_i)L_a\text{pos}(m_j)$  holds for two attribute concepts, we have  $\varphi(\text{pos}(m_i)) < \varphi(\text{pos}(m_j))$  and  $\text{pos}(\mu m_i)L\text{pos}(\mu m_j)$  respectively.

By lemma 10 and 5 we conclude that  $L(\text{pos}(\underline{\mathfrak{B}}(G, M, I)))$  is the corresponding  $L$ -relation to  $L_a$ . Additionally, we showed that  $L$  is a strict order.

## 4 Results and Further Work

We have shown in this work, that the relationship of attribute concepts indeed provides an instrument to characterize, whether the associated concept lattice is planar. Unfortunately an efficient algorithm using that connection is not developed yet. Such an algorithm would be very useful for a lattice layout program using the attribute (or dually object) additive convention for quick planarity testing and embedding. The next steps to do are:

- Find a fast algorithm to compute all planar sorting relations.
- Classify plane diagrams of a lattice up to homeomorphisms.
- Proof that every planar lattice possesses a plane attribute additive diagram.

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