Defining Finitely Supported Mathematics over Sets with Atoms

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Abstract. This paper presents some steps of defining a finitely supported mathematics by using sets with atoms. Such a mathematics generalizes the classical Zermelo-Fraenkel mathematics, and represents an appropriate framework to work with (infinite) structures in terms of finitely supported objects. We focus on the techniques of translating the Zermelo-Fraenkel results to this finitely supported mathematics over sets with atoms.

Keywords: Fraenkel-Mostowski set theory, invariant sets, finite support principle, Finitely Supported Mathematics.

Key-Terms: FormalMethod, MathematicalModel, Research.

1 Introduction

Since the experimental sciences are mainly interested in quantitative aspects, and since there exists no evidence for the presence of infinite structures, it becomes useful to develop a mathematics which deals with a more relaxed notion of (in)finiteness. We present our attempt of building the necessary concepts and structures for a finitely supported mathematics. What we call Finitely Supported Mathematics is a mathematics which is consistent with the axioms of the Fraenkel-Mostowski (FM) set theory. The FM axioms represents an "axiomatization" of the FM permutation model of the Zermelo-Fraenkel set theory with atoms; in this way, these axioms transform this model into an independent set theory. The axioms of the FM set theory are precisely the Zermelo-Fraenkel with atoms (ZFA) axioms over an infinite set of atoms [16], together with the special property of finite support which claims that for each element x in an arbitrary set we can find a finite set supporting x. Therefore in the FM universe only finitely supported objects are allowed. The original purpose of the FM set theory was to provide a mathematical model for variables in a certain syntax. Since they have no internal structure, atoms can be used to represent names. The finite support axiom is motivated by the fact that syntax can only involve finitely many names. The FM set theory provides a balance between rigorous formalism and informal reasoning. This is discussed in [23], where principles of structural recursion and induction are explained in the FM framework. We can use this theory in order to manage infinite structures in a finitary manner, that is, in the FM framework we try to model the infinite using a more relaxed notion of finite, i.e, the notion of finite support.

Although a set of axioms for describing sets with atoms (or FM-sets) was introduced in [16], an earlier idea of using atoms in computer science belongs to Gandy [17]. Gandy proved that any machine satisfying four physical 'principles' is equivalent to some Turing machine. Gandy's four principles define a class of computing machines, namely the 'Gandy machines'. Gandy machines are represented by classes of 'states' and 'transition operations between states'. States are represented by hereditary finite sets built up from an infinite set U of atoms, and transformations are given by restricted operations from states to states. The class HF of all hereditary finite sets over U introduced in Definition 2.1 from [17] is described quite similar to the von-Neumann cumulative hierarchy of FM-sets, FM_A presented in [16]. The single difference between these approaches is that each HF_{n+1} is defined inductively involving '*finite* subsets of $U \cup HF_n$', whilst each $FM_{\alpha+1}(A)$ is defined inductively by using 'the disjoint union between A and the *finitely supported* subsets of $FM_{\alpha}(A)$'; HF is the union of all HF_n (with the mention that the empty set is not used in this construction), and the family of all FM-sets is the union of all FM_{α} from which we exclude the set A of atoms. The support of an element x in HF, obtained according to Definition 2.2(1) of [17], coincides with $supp(x)$ (with notations from Definition 2(4)) if we see x as an FM-set. Also, the effect of a permutation π on a structure x described in Definition 2.3 from [17] is defined analogue as the application of the S_A-action on FM_A to the element $(\pi, x) \in S_A \times FM_A$. Obviously, the Gandy's principles can also be presented in the FM framework because any finite set is well defined in FM; however, an open problem regards the consistency of Gandy's principles when 'finite' is replaced by 'finitely supported'.

The construction of the universe of all FM-sets [16] is inspired by the construction of the universe of all admissible sets over an arbitrary collection of atoms [6]. The hereditary finite sets used in [17] are particular examples of admissible sets. The FM-sets represent a generalization of hereditary finite sets because any FM-set is an hereditary finitely supported set.

In the literature there exist various approaches regarding the FM framework. We try to clarify the differences between these approaches.

- **The FM permutation model of the ZFA set theory.**
  This model was introduced by Fraenkel [14] and extended by Lindenbaum and Mostowski [21]. Its original aim was to establish the independence of the axiom of choice from the other axioms of the ZFA set theory. There also exist some other permutation models of ZFA presented in [20] which are defined by using countable infinite sets of atoms.
- **The FM axiomatic set theory.** This set theory was presented in [16]. It is inspired by the FM permutation model of the ZFA set theory. However, the FM set theory, the ZFA set theory and the Zermelo-Fraenkel (ZF) set theory are independent axiomatic set theories. All of these theories are described by

axioms, and all of them have models. For example, the Cumulative Hierarchy Fraenkel-Mostowski universe FM_A presented in [16] is a model of the FM set theory, while some models of the ZF set theory can be found in [19], and the permutation models of the ZFA set theory can be found in [20]. The sets defined using the FM axioms are called FM-sets. A ZFA set is an FM-set if and only if all its elements have hereditarily finite supports. Note that the infinite set of atoms in the FM set theory does not necessary be countable. The Fraenkel-Mostowski set theory is consistent whether the infinite set of atoms is countable or not. In [16] it is used a countable set of atoms in order to define a model of the Fraenkel-Mostowski set theory for new names in computer science, while in [7] there are described FM-sets over a set of atoms which do not represent a homogeneous structure. Also, in [12] the authors use non-countable sets of atoms (like the set of real numbers) in order to study the minimization of deterministic timed automata.
- **Nominal sets.** These sets can be defined both in the ZF framework [24] and in the FM framework [16]. In ZF, a fixed infinite set A is considered as a set of names. A nominal set is defined as a usual ZF set endowed with a particular group action of the group of permutations over A that satisfies a certain finiteness property. Such a finiteness property allows us to say that nominal sets are well defined according to the axioms of the FM set theory whenever the set of names is the set of atoms in the FM set theory. There exists also an alternative definition for nominal sets in the FM framework. They can be defined as sets constructed according to the FM axioms with the additional property of being empty supported (invariant under all permutations). These two ways of defining nominal sets finally lead to similar properties. According to the previous remark we use the terminology "invariant" for "nominal" in order to establish a connection between approaches in the FM framework and in the ZF framework. Moreover, we can say that any set defined according to the FM axioms (any FM-set) can be seen as a subset of the nominal (invariant) set FM_A. However, an FM-set is itself a nominal set only if it has an empty support. The theory of nominal sets makes sense even if the set of atoms is infinite but not countable. Informally, since the ZFA set theory collapses into the ZF set theory when the set of atoms is empty, we can say that the nominal sets represent a natural extension of the usual sets. In computer science, nominal sets offer an elegant formalism for describing λ-terms modulo α-conversion [16]. They can also be used in algebra [5, 2], in proof theory [27], in domain theory [26], in topology [22], semantics of process algebras [4, 15] and programming [25]. A survey on the applications of nominal sets in computer science emphasizing our contributions can be found in [3].
- **Generalized nominal sets.** The theory of nominal sets over a fixed set A of atoms is generalized in [10] to a new theory of nominal sets over arbitrary (unfixed) sets of data values. This provides the generalized nominal sets. The notion of 'S_A-set' (Definition 2) is replaced by the notion of 'set endowed with an action of a subgroup of the symmetric group of $\mathbb{D}$' for an arbitrary set of data values $\mathbb{D}$, and the notion of 'finite set' is replaced by the notion of 'set

with a finite number of orbits according to the previous group action (orbit-finite set)'. This approach is useful for studying automata on data words [10], languages over infinite alphabets [8], or Turing machines that operate over infinite alphabets [11]. Computations in these generalized nominal sets are presented in [9, 13].

As their names say, the nominal sets are used to manage notions like renaming, binding or fresh name. However, this theory could be studied deeper from an algebraically viewpoint, and it could be used in order to characterize some infinite structures in terms of finitely supported objects.

Finitely Supported Mathematics (FSM) is introduced to prove that many finiteness ZF properties still remain valid if we replace the term 'finite' with 'infinite, but with finite support'. Such results have already been presented in [5] where we proved that a class of multisets over infinite alphabets (interpreted in the nominal framework) has similar properties to the classical multisets over finite alphabets. FSM is the mathematics developed in the world of finitely supported objects where the set of atoms has to be infinite (countable or not countable). Informally, FSM extends the framework of the ZF set theory without choice principles; ZF set theory is actually the Empty Supported Mathematics. In FSM, we use either 'invariant sets' or 'finitely supported sets' instead of 'sets'. As an intuitive rule, we are not allowed to use in the proofs of the results of FSM any construction that does not preserve the property of finite support. That means we cannot obtain a property in FSM only by using a ZF result without an appropriate proof using only the finite support condition. Since the invariant sets can also be defined in the ZFA framework similarly as in the ZF framework (see the first paragraph in Section 2), the definition of the finitely supported mathematics also makes sense over the ZFA axioms.

To summarize, FSM represents the ZF theory rephrased in terms of finitely supported objects; this means that FSM presents the theory of invariant sets, including invariant algebraic structures. FSM is not at all the theory of nominal sets from [24] presented in a different manner; actually the theory of nominal sets [24] could be considered as a tool for defining FSM. The main aim of FSM is to characterize the infinite algebraic structures by using their finite supports.

2 Sets with Atoms

Let A be a fixed infinite (countable or non-countable) ZF-set. The following results make also sense if A is considered to be the set of atoms in the ZFA framework (characterized by the axiom "$y \in x \Rightarrow x \notin A$") and if 'ZF' is replaced by 'ZFA' in their statements. Thus, we mention that the theory of invariant sets makes sense both in ZF and in ZFA. Several results of this section are similar to those in [24], but without assuming the set of atoms to be countable.

Definition 1. *A* transposition *is a function* $(a\,b) : A \to A$ *defined by* $(a\,b)(a) = b$, $(a\,b)(b) = a$, *and* $(a\,b)(n) = n$ *for* $n \neq a, b$. *A* permutation *of* A *is generated by composing finitely many transpositions.*

Definition 2. *Let S_A be the set of all permutations of A.*

1. *Let X be a ZF set. An S_A-*action *on X is a function $\cdot : S_A \times X \to X$ having the properties that $Id \cdot x = x$ and $\pi \cdot (\pi' \cdot x) = (\pi \circ \pi') \cdot x$ for all $\pi, \pi' \in S_A$ and $x \in X$. An S_A-*set *is a pair $(X, \cdot)$ where X is a ZF set, and $\cdot : S_A \times X \to X$ is an S_A-action on X.*
2. *Let $(X, \cdot)$ be an S_A-set. We say that $S \subset A$* supports *x whenever for each $\pi \in Fix(S)$ we have $\pi \cdot x = x$, where $Fix(S) = \{\pi \,|\, \pi(a) = a, \forall a \in S\}$.*
3. *Let $(X, \cdot)$ be an S_A-set. We say that X is an* invariant set *if for each $x \in X$ there exists a finite set $S_x \subset A$ which supports x. Invariant sets are also called* nominal sets *if we work in the ZF framework [24], or* equivariant sets *if they are defined as elements in the cumulative hierarchy FM_A [16].*
4. *Let X be an S_A-set and let $x \in X$. If there exists a finite set supporting x, then there exists a least finite set supporting x [16] which is called* the support of *x and is denoted by $supp(x)$. An element supported by the empty set is called equivariant.*

Proposition 1. *Let $(X, \cdot)$ be an S_A-set and $\pi \in S_A$. If $x \in X$ is finitely supported, then $\pi \cdot x$ is finitely supported, and $supp(\pi \cdot x) = \pi(supp(x))$.*

Example 1.

1. The set A of atoms is an S_A-set with the S_A-action $\cdot : S_A \times A \to A$ defined by $\pi \cdot a := \pi(a)$, $\forall \pi \in S_A, a \in A$. Moreover, $supp(B) = B$, $\forall B \subset A$, B finite.
2. Any ordinary ZF set X (like $\mathbb{N}$ or $\mathbb{Z}$) is an S_A-set with the trivial S_A-action $\cdot : S_A \times X \to X$ defined by $\pi \cdot x := x$ for all $\pi \in S_A$ and $x \in X$.
3. If $(X, \cdot)$ is an S_A-set, then $\wp(X) = \{Y \,|\, Y \subseteq X\}$ is also an S_A-set with the S_A-action $\star : S_A \times \wp(X) \to \wp(X)$ defined by $\pi \star Y := \{\pi \cdot y \,|\, y \in Y\}$ for all $\pi \in S_A$, and all subsets Y of X. For each invariant set $(X, \cdot)$ we denote by $\wp_{fs}(X)$ the set formed from those subsets of X which are finitely supported according to the action $\star$. $(\wp_{fs}(X), \star|_{\wp_{fs}(X)})$ is an invariant set, where $\star|_{\wp_{fs}(X)}$ represents the action $\star$ restricted to $\wp_{fs}(X)$.
4. Let $(X, \cdot)$ and $(Y, \diamond)$ be S_A-sets. The Cartesian product $X \times Y$ is also an S_A-set with the S_A-action $\star : S_A \times (X \times Y) \to (X \times Y)$ defined by $\pi \star (x, y) = (\pi \cdot x, \pi \diamond y)$ for all $\pi \in S_A$ and all $x \in X$, $y \in Y$. If $(X, \cdot)$ and $(Y, \diamond)$ are invariant sets, then $(X \times Y, \star)$ is also an invariant set.
5. The FM cumulative hierarchy FM_A described in [16] is an invariant set with S_A-action $\cdot : S_A \times FM_A \to FM_A$ defined inductively by $\pi \cdot a := \pi(a)$ for all atoms $a \in A$ and $\pi \cdot x := \{\pi \cdot y \,|\, y \in x\}$ for all $x \in FM_A \setminus A$. An FM-set is a finitely supported element in FM_A; additionally an FM-set has the recursive property that all its elements are also FM-sets. An FM-set which is empty supported as an element in FM_A is an invariant set.

Definition 3. *Let $(X, \cdot)$ be an invariant set. A subset Z of X is called* finitely supported *if and only if $Z \in \wp_{fs}(X)$ with the notations of Example 1 (3).*

Definition 4. *Let X and Y be invariant sets, and let Z be a finitely supported subset of X. A function $f : Z \to Y$ is* finitely supported *if $f \in \wp_{fs}(X \times Y)$.*

Proposition 2. *[5] Let $(X, \cdot)$ and $(Y, \diamond)$ be invariant sets, and let Z be a finitely supported subset of X. The function $f : Z \to Y$ is finitely supported in the sense of Definition 4 if and only if there exists a finite set S of atoms such that for all $x \in Z$ and all $\pi \in Fix(S)$ we have $\pi \cdot x \in Z$ and $f(\pi \cdot x) = \pi \diamond f(x)$.*

3 Reformulating the Classical ZF Results in FSM

The main idea of translating a classical ZF result (depending on sets and relations) into FSM is to analyze if there exists a valid result obtained by replacing "set" with "invariant/finitely supported set" and "relation" with "invariant/finitely supported relation" in the ZF result. If this is possible, then things go smoothly; however, this is not always so simple.

Every ZF set is a particular invariant set equipped with a trivial permutation action (Example 1(2)). Therefore, the general properties of invariant sets lead to valid properties of ZF sets. The converse is not always valid, namely not every ZF result can be directly rephrased in the world of invariant sets, terms of finitely supported objects according to arbitrary permutation actions. This is because, given an invariant set X, there could exist some subsets of X (and also some relations or functions involving subsets of X) which fail to be finitely supported. A classical example (presented also in Subsection 2.2.3.6 of [26]) is represented by the powerset of the invariant set A. A subset of A which is in the same time infinite and coinfinite could be defined in some models of ZF (or of ZFA is we consider A to be the set of atoms in ZFA), but it can not be defined in FSM because it is not finitely supported. Therefore the remark that everything that can be done in ZF can also be done in FSM is not valid. That means there may exist some valid results depending on several ZF structures which fail to be valid in FSM if we simply replace "ZF structure" with "FSM structure" in their statement.

We present few examples regarding these aspects. There exist some valid ZF results that cannot be translated into FSM. According to Remark 1, the following examples are particularly interesting because they do not overlap neither on some known properties of permutative models of ZFA, nor on some properties of nominal sets [24].

Example 2.

- There exist models of ZF without choice that satisfy the ordering principle "Every set can be totally ordered". More details about such models are in [19], where there are mentioned Howard-Rubin's first model N38 and Cohen's first model M1. Therefore the ordering principle is independent from the axioms of the ZF set theory.
- In FSM the following result fails "For every invariant set X there exists a finitely supported total order relation on X". Therefore the ordering principle is inconsistent with the axioms of the FM set theory. Indeed, suppose that there exists a finitely supported total order $<$ on the invariant set A. Let $a, b, c \notin supp(<)$ with $a < b$. Since $(a\, c) \in Fix(supp(<))$ we have $(a\, c)(a) <$

$(a\,c)(b)$, so $c < b$. However, we also have $(a\,b), (b\,c) \in Fix(supp(<))$, and so $((a\,b) \circ (b\,c))(a) < ((a\,b) \circ (b\,c))(b)$, that is, $b < c$. We get a contradiction, and so the translation of the ordering principle in FSM realized by replacing "structure" with "finitely supported structure" leads to a false statement.

Example 3.

- There exist models of ZF without choice that satisfy the partial countable choice principle: "Given any countable family (sequence) of non-empty sets $\mathcal{F} = (X_n)_n$, there exists an infinite subset M of $\mathbb{N}$ such that it is possible to select a single element from each member of the family $(X_m)_{m \in M}$, i.e. there exist a choice function on $(X_m)_{m \in M}$". More details about such models are in [19], where there are mentioned Pincus-Solovay's First Model M27, Shelah's Second Model M38 and Howard-Rubin's first model N38. Therefore the partial countable choice principle is independent from the axioms of the ZF set theory.
- In FSM the following result fails: "Given any invariant set X, and any countable family $\mathcal{F} = (X_n)_n$ of subsets of X such that the mapping $n \mapsto X_n$ is finitely supported, there exists an infinite subset M of $\mathbb{N}$ with the property that there is a finitely supported choice function on $(X_m)_{m \in M}$". Therefore the partial countable choice principle is inconsistent with the axioms of the FM set theory. Indeed, for the invariant set A we consider the countable family $(X_n)_n$ where X_n is the set of all injective n-tuples from A. Since A is infinite, it follows that each X_n is non-empty. In the FM framework, each X_n is equivariant because A is an invariant set and each permutation is a bijective function. Therefore the family $(X_n)_n$ is equivariant, and the mapping $n \mapsto X_n$ is also equivariant. Suppose that there exists an infinite subset M of $\mathbb{N}$ and a finitely supported choice function f on $(X_m)_{m \in M}$. Let $f(X_m) = y_m$ with each $y_m \in X_m$. Let $\pi \in Fix(supp(f))$. According to Proposition 2, and because each element X_m is equivariant according to its definition, we obtain that $\pi \cdot y_m = \pi \cdot f(X_m) = f(\pi \cdot X_m) = f(X_m) = y_m$. Therefore, each element y_m is supported by $supp(f)$, and so $supp(y_m) \subseteq supp(f)$ for all $m \in M$. Since y_m is a finite tuple of atoms which has exactly m elements for each $m \in M$, we have that $supp(y_m) = y_m$, $\forall m \in \mathbb{N}$ (see Example 1(1)). Thus $y_m \subseteq supp(f)$ for all $m \in M$. However, because M is infinite, we contradict the finiteness of $supp(f)$. Therefore the translation of the partial countable choice principle in FSM realized by replacing "structure" with "finitely supported structure" leads to a false statement.

Remark 1. Examples 2 and 3 show us that there exist some choice principles which are independent from the axioms of the ZF set theory, but inconsistent in FSM. Since FSM is consistent even if the set of atoms is not countable, such results do not overlap on some related properties in the basic or in the second Fraenkel modes of the ZFA set theory (which are defined using countable sets of atoms) [20]. Also, the previous results do not follow immediately from [24] because the nominal sets are defined over countable sets of atoms, while we define invariant sets over possible non-countable sets of atoms; in [24] where the set of atoms is countable, Example 3 would be trivial. Moreover, we claim that all the choice principles from [18] rephrased in terms of invariant sets are inconsistent in FSM. Note that it is not easy to prove such a result in FSM, even if various relationship results between several forms of choice hold in the ZF framework. This is because nobody guarantees that ZF results remain valid in FSM. Therefore, all the possible relationship results between various choice principles in FSM have to be independently proved in terms of finitely supported object. Details regarding the consistency of various choice principles in the world of invariant sets defined over possibly non-countable sets of atoms are presented in another paper.

Other results which fail in FSM are given by the Stone duality [22], by the determinization of finite automata and by the equivalence of two-way and one-way finite automata [10]. There also exist some valid ZF results that can be translated into FSM only in a weaker form.

Example 4. We define an invariant complete lattice as an invariant set $(L, \cdot)$ together with an equivariant order relation $\sqsubseteq$ on L satisfying the property that every finitely supported subset $X \subseteq L$ has a least upper bound with respect to the order relation $\sqsubseteq$.

- Let L be a ZF complete lattice and $f : L \to L$ a ZF monotone function. Then there exists a greatest $e \in L$ such that $f(e) = e$ and a least $e \in L$ such that $f(e) = e$ (weak form of Tarski theorem).
- Let $(L, \sqsubseteq, \cdot)$ be an invariant complete lattice and $f : L \to L$ a finitely supported monotone function. Then there exists a greatest $e \in L$ such that $f(e) = e$, and a least $e \in L$ such that $f(e) = e$ (the proof is similar to Theorem 3.2 in [1]).

These results show that the weak form of the Tarski theorem can be naturally translated into FSM. However, as it is presented below, the strong form of the Tarski theorem cannot be naturally translated into FSM; it holds in FSM only for a particular class of finitely supported monotone functions, i.e, the equivariant monotone functions.

- Let L be a ZF complete lattice and $f : L \to L$ a ZF monotone function over L. Let P be the set of fixed points of f. Then P is a complete lattice (strong form of Tarski theorem).
- Let $(L, \sqsubseteq, \cdot)$ be an invariant complete lattice and $f : L \to L$ an equivariant monotone function over L. Let P be the set of fixed points of f. Then $(P, \sqsubseteq, \cdot)$ is an invariant complete lattice.
  The result does not hold if f is finitely supported, but not equivariant (the proof is similar to Theorem 3.3 in [1]).

4 Limits of the Equivariance / Finite Support Principle

In order to translate a general ZF result into FSM, one must prove that several structures are finitely supported. There exist two general methods of proving that a certain structure is finitely supported. The first method is a constructive one: by using some intuitive arguments, we anticipate a possible candidate for the support and prove that this candidate is indeed a support. The second method is based on a general finite support principle which is defined using the higher-order logic. However the use of this second method has some limits, as we present in the paragraphs below.

According to [23], we have the following equivariance/finite support principle which works over invariant sets.

Theorem 1.

- *Any function or relation that is defined from equivariant functions and relations using classical higher-order logic is itself equivariant.*
- *Any function or relation that is defined from finitely supported functions and relations using classical higher-order logic is itself finitely supported.*

In applying this equivariance/finite support principle, one must take into account all the parameters upon which a particular construction depends. We think that the formal involvement of the equivariance/finite support principle, i.e. the precise verification if the conditions for applying the equivariance/finite support principle are properly satisfied is sometimes at least as difficult as a constructive proof. Moreover, in many cases we need to construct effectively the support, and it is not enough to prove only that a certain structure is finitely supported.

Example 5. An invariant monoid $(M, \cdot, \diamond)$ is an invariant set $(M, \diamond)$ endowed with an equivariant internal monoid law $\cdot : M \times M \to M$. If $(\Sigma, \diamond)$ is an invariant set, then the free monoid Σ^* on Σ is an invariant monoid [5].

1. For each monoid M and each function $f : \Sigma \to M$, there exists a unique homomorphism of monoids $g : \Sigma^* \to M$ with $g \circ i = f$, where $i : \Sigma \to \Sigma^*$ is the standard inclusion of Σ into Σ^* which maps each element $a \in \Sigma$ into the word a (ZF universality theorem for monoids).
2. i) Let $(\Sigma, \diamond_\Sigma)$ be an invariant set. Let $i : \Sigma \to \Sigma^*$ be the standard inclusion of Σ into Σ^* which maps each element $a \in \Sigma$ into word a. If $(M, \cdot, \diamond_M)$ is an arbitrary invariant monoid and $\varphi : \Sigma \to M$ is an arbitrary finitely supported function, then there exists a unique finitely supported homomorphism of monoids $\psi : \Sigma^* \to M$ with $\psi \circ i = \varphi$.
   This result can be proved directly by involving the equivariance/finite support principle.

   ii) Let $(\Sigma, \diamond_\Sigma)$ be an invariant set. Let $i : \Sigma \to \Sigma^*$ be the standard inclusion of Σ into Σ^* which maps each element $a \in \Sigma$ into the word a. If $(M, \cdot, \diamond_M)$ is an arbitrary invariant monoid and $\varphi : \Sigma \to M$ is an arbitrary finitely supported function, then there exists a unique finitely supported homomorphism of monoids $\psi : \Sigma^* \to M$ with $\psi \circ i = \varphi$. Moreover, if a finite set S supports φ, then the same set S supports ψ.
   The last sentence of this theorem cannot be proved by involving the equivariance/finite support principle.

Proof. If $(M, \cdot, \diamond_M)$ is an invariant monoid, then $(M, \cdot)$ is a monoid. From the general ZF theory of monoids, we can define a unique homomorphism of monoids $\psi : \Sigma^* \to M$ with $\psi \circ i = \varphi$.

In [5] we proved that the free monoid Σ^* on Σ is an invariant monoid whenever $(\Sigma, \diamond)$ is an invariant set. The S_A-action $\widetilde{\star} : S_A \times \Sigma^* \to \Sigma^*$ is defined by $\pi \widetilde{\star} x_1 x_2 \dots x_l = (\pi \diamond x_1) \dots (\pi \diamond x_l)$ for all $\pi \in S_A$ and $x_1 x_2 \dots x_l \in \Sigma^* \setminus \{1\}$, and $\pi \widetilde{\star} 1 = 1$ for all $\pi \in S_A$.

In order to prove that ψ is finitely supported it is sufficient to apply Theorem 1 because ψ is defined from the finitely supported functions φ and i using the higher-order logic. However, Theorem 1 is not sufficient to prove that if a finite set S supports φ, then the same set S supports ψ. In order to prove the previous statement we proceed as follows.

Let us consider $S = supp(\varphi)$. Thus, by Proposition 2 we have $\varphi(\pi \diamond_\Sigma x) = \pi \diamond_M \varphi(x)$ for all $x \in \Sigma$ and $\pi \in Fix(S)$. We have to prove that S supports ψ. Let $\pi \in Fix(S)$. According to Proposition 2 it is sufficient to prove that $\psi(\pi \widetilde{\star} x_1 x_2 \dots x_n) = \pi \diamond_M \psi(x_1 x_2 \dots x_n)$ for each $x_1 x_2 \dots x_n \in \Sigma^*$. However, ψ is a monoid homomorphism between Σ^* and M, and $\psi \circ i = \varphi$. This means $\psi(x_1 x_2 \dots x_n) = \varphi(x_1) \cdot \varphi(x_2) \cdot \dots \cdot \varphi(x_n)$. Since $(M, \cdot, \diamond_M)$ is an invariant monoid we have $\pi \diamond_M \psi(x_1 x_2 \dots x_n) = \pi \diamond_M (\varphi(x_1) \cdot \varphi(x_2) \cdot \dots \cdot \varphi(x_n)) = (\pi \diamond_M \varphi(x_1)) \cdot (\pi \diamond_M \varphi(x_2)) \cdot \dots \cdot (\pi \diamond_M \varphi(x_n)) = \varphi(\pi \diamond_\Sigma x_1) \cdot \varphi(\pi \diamond_\Sigma x_2) \cdot \dots \cdot \varphi(\pi \diamond_\Sigma x_n)$. However, $\pi \widetilde{\star} x_1 x_2 \dots x_n = (\pi \diamond_\Sigma x_1) \dots (\pi \diamond_\Sigma x_n)$ and $\psi(\pi \widetilde{\star} x_1 x_2 \dots x_n) = \psi((\pi \diamond_\Sigma x_1) \dots (\pi \diamond_\Sigma x_l)) = \varphi(\pi \diamond_\Sigma x_1) \cdot \varphi(\pi \diamond_\Sigma x_2) \cdot \dots \cdot \varphi(\pi \diamond_\Sigma x_n)$. Hence $\psi(\pi \widetilde{\star} x_1 x_2 \dots x_n) = \pi \diamond_M \psi(x_1 x_2 \dots x_n)$ for each $\pi \in Fix(S)$, which means S supports ψ.

Example 5(2) shows us that by using the equivariance/finite support principle we can obtain a universality property for invariant monoids which is similar to the one described in Example 5(1). However, in order to prove that $supp(\psi) \subseteq supp(\varphi)$ in the second item of Example 5(2), we need to present a constructive method of defining a set supporting ψ (see also Theorem 6 from [5]). Other related examples regarding the equivariance/finite support principle are Theorems 4, 9 and 11 from [5], or Theorem 3.7 from [2]. In these theorems we are able to prove a precise characterization for the support of some structures which could not be obtained by a direct application of the equivariance/finite support principle in the form from Theorem 1. In these results we do not prove only that some structures are finitely supported, but we also found a relationship between the supports of the related structures.

In some cases we can prove stronger properties without involving the equivariance/finite support principle. For example, each function f_x in the proof of Theorem 7 of [5] has a non-empty finite support. Using the equivariance/finite support principle one can say that the function T from that theorem has also a finite support. We were able to prove something stronger using a constructive method: the function T is equivariant.

A *constructive method* of defining the support is also necessary in order to assure that some structures are uniformly finitely supported (i.e. supported by the same finite set of atoms). Some related examples regarding the uniform support are presented in [2] (Section 5), where we should assume that some structures are uniformly supported in order to obtain some embedding properties for invariant (nominal) groups. Also, note that a chain is finitely supported if and only if all its elements are finitely supported and have the same support, i.e., all its elements are uniformly finitely supported. Therefore, in order to prove that a chain is finitely supported, we must present a constructive method of defining the support of its elements. More exactly, we cannot use the equivariance/finite support principle which would not assure the uniformity of the support of its elements. Suggestive examples regarding finitely supported chains are presented in Chapter 4 of [25].

We conclude that the equivariance/finite support principle is not useful when we want to obtain a relationship between the supports of several constructions (and we do not want only to prove that these constructions are finitely supported). This is because, in its actual form, the second part of Theorem 1 allows to prove that a certain structure is finitely supported, but it do not provide any information about the structure of the support. However, the first part of Theorem 1 helps when we want to prove the equivariance of some constructions. Note that we do not claim that the finite support principle is not useful. Obviously, it can be used to give simpler proofs for the fact that functions and relations defined from finitely supported functions and relations via classical higher-order formulas are finitely supported. However, a concrete calculation for the supports of some structures is able to provide more informations about the related supports; we justify this viewpoint in Example 5. Also, such a method is useful in order to find the uniform supports.

Note that, often in practice, it is not sufficient to prove only that a certain structure is finitely supported without giving any information about the structure of support. A more precise characterization of the support is useful. For example, let us consider an α-equivalence class $[t]$ of a λ-term t. The support of $[t]$ is represented by the set of free names of t [16]; the support of $[t]$ is finite because any λ-term has a finite numbers of free names. However, the precise description of the free names of t is an aspect that matters. Therefore, we suggest to use a constructive method of defining the support of a certain structure instead of the finite support part (the second part) of Theorem 1, because in this way we can obtain more informations about the support.

5 Conclusion

Our goal is to develop a mathematics for experimental science which deals with a more relaxed notion of finiteness. We call it the 'Finitely Supported Mathematics'. Informally, in Finitely Supported Mathematics we can model infinite structures after a finite number of observations. More precisely, we intend to restate some parts of algebra by replacing '(infinite) sets' with 'invariant sets'. This allows to model some infinite structures by using their finite supports. In order to sustain our viewpoint, we involve the axiomatic theory of FM-sets presented in [16]. Rather than using a non-standard set theory, we could alternatively work with invariant sets, which are defined within ZF as usual sets endowed with some group actions satisfying a finite support requirement. The properties of invariant sets are similar to those presented in [24], with the mention that we assume invariant sets to be defined over possible non-countable sets of atoms. Our paper presents the basic steps requested in order to provide an extension of the theory of invariant sets to a theory of invariant algebraic structures. Although the initial purpose of defining invariant sets was to formulate a semantics for syntax with variable binding, we consider that such sets can also be used from an algebraic perspective in order to characterize infinite structures modulo finite supports, and thus in order to provide more informations about infinite objects.

The category of invariant sets has a very rich structure, and so the definitions of many structures given in the usual category of sets can be reformulated within the invariant sets framework. A natural question is which classical theorems about these structures hold internally in the world of invariant sets. Until now (or, more precisely, until we would be able to solve the open problem presented below), there does not exist a standard algorithm to translate any classical ZF result into FSM. This is because there may exist some subsets of an invariant set which fail to be finitely supported, and thus there may exist some ZF results that fail in the universe of invariant sets. Related examples regarding the previous statement are presented in Section 3. Therefore, reformulating the ZF theorems into FSM should be done for each case separately. For example, the theory of monoids is studied in FSM in [5], the theory of groups is rephrased in FSM in [2], and the theory of posets and domains is reformulated within invariant sets framework in [24, 25]. In order to prove that a structure is finitely supported, one could use either the finite support principle of [24] (e.g. Theorem 1), or a more "constructive" method. To employ such a "constructive method" means that we anticipate a possible candidate for a support, and then prove that this candidate is indeed a support. The benefit of this method is that we are able to obtain more informations about the related support than by using the finite support principle. Related examples can be found in Section 4.

An Open Problem: The main task in order to define a finitely supported mathematics is to prove that certain subsets of an invariant set are finitely supported. We already know that given an invariant set X, there could exist some subsets of X which fail to be finitely supported. Some related examples are presented in [24] and [26]. However, all these examples are described by using choice principles or consequences of choice principles (like the assertion that the set A can be non-amorphous in ZFA) in order to construct some structures which later fail to be finitely supported. We conjecture that all the choice principles presented in [18] are inconsistent in FSM. We did not find yet any example of a non-finitely supported subset of an invariant set defined without using a choice principle from [18] or a consequence of a form of choice (like the construction of an infinite and coinfinite subset of an infinite set). Therefore, the question regarding the validity of the following assertions naturally appears.

- If we consider the ZF set theory (or the ZFA set theory) without any choice principle, then every subset of an invariant set is finitely supported?

– For what kind of atoms the previous question has an affirmative answer?

If we get an affirmative answer (even for a particular set of atoms), then the mathematics developed in the ZF (or ZFA) set theory without any choice principle would be somehow equivalent to FSM, namely we could model any infinite structure by using its finite support.

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