Optimization Problems on Posets

Christian Jäkel[1] and Stefan E. Schmidt[2]

Technische Universität Dresden, 01062, Dresden, Germany

[1]ChristianJaekel(at)gmx-topmail.de
[2]midt1(at)msn.com

Abstract. This article treats four optimization problems on posets and applies the results to Formal Concept Analysis. The cover-, packing-, isolation- and blocking problem are investigated.

Keywords: formal concept analysis, poset, order dimension, cover problem, packing problem, isolated set, blocking set, cardinal product.

1 Introduction

In [6] the *rectangle cover number* was introduced as the minimal number of maximal rectangles which are necessary to cover the incidence relation of a formal context. The formal definition is:

$$\mathrm{rc}(\mathbb{K}) := \min\{\#\mathcal{R} \mid \mathcal{R} \subseteq \mathrm{Rec}(\mathbb{K}),\ I = \bigcup_{R \in \mathcal{R}} R\}.$$

This optimization problem on formal contexts can be translated to the order 2-dimension of the concept lattice of the complementary context (see [5]). As a lower bound for the rectangle cover number, the *rectangle isolation number* was introduced in [6] as the maximal number of elements from I, such that every maximal rectangle contains at most one of them:

$$\mathrm{ri}(\mathbb{K}) := \max\{\#\mathcal{I} \mid \mathcal{I} \subseteq I,\ \forall(R \in \mathrm{Rec}(\mathbb{K})) : \#(\mathcal{I} \cap R) \leq 1\}.$$

It always holds that $\mathrm{ri}(\mathbb{K}) \leq \mathrm{rc}(\mathbb{K})$ and in case of equality there are consequences for the tensor product of complete lattices (see [6]). After investigating these optimization problems for a while, two seemingly similar problems come up. These are the maximal number of disjoint rectangles that fit into I, denoted as the *rectangle packing number*, and the minimal number of elements from I such that is at least one of them contained in every maximal rectangle, denoted as the *rectangle blocking number*. Formally:

$$\mathrm{rp}(\mathbb{K}) := \max\{\#\mathcal{R} \mid \mathcal{R} \subseteq \mathrm{Rec}(\mathbb{K}),\ \forall(R_1, R_2 \in \mathcal{R}),\ R_1 \neq R_2 : R_1 \cap R_2 = \emptyset\},$$
$$\mathrm{rb}(\mathbb{K}) := \min\{\#\mathcal{I} \mid \mathcal{I} \subseteq I,\ \forall(R \in \mathrm{Rec}(\mathbb{K}))\,\exists((g,m) \in \mathcal{I}) : (g,m) \in R\}.$$

Similar, to the cover and isolation number, it holds that $\mathrm{rp}(\mathbb{K}) \leq \mathrm{rb}(\mathbb{K})$.

This paper treats the above defined optimization problems on formal contexts. Therefore, Section 2 provides necessary definitions from formal concept analysis. Section 3 analyses the optimization problems and their mutual relationships. This is done in a more general setting based on posets. This abstraction is the main content of this paper in it's own right. Concrete examples from Formal Concept Analysis will be presented along the way. The last section provides a conclusion.

2   Basics of Formal Concept Analysis

In this section, we will provide the facts from formal concept analysis that we will use in the sequel. All results can be found in [5].

A *formal context* is a triple $\mathbb{K} = (G, M, I)$, where the *incidence* $I \subseteq G \times M$ is a binary relation. For $A \subseteq G$ and $B \subseteq M$, we define two derivation operators:

$$A^I := \{m \in M \mid \forall (a \in A) : (a, m) \in I\} = \bigcap_{a \in A} \{a\}^I,$$

$$B_I := \{g \in G \mid \forall (b \in B) : (g, b) \in I\} = \bigcap_{b \in B} \{b\}_I.$$

If $A^I = B$ and $B_I = A$, the pair (A, B) is called a *formal concept* and for non empty A and B the cartesian product $A \times B$ is a *maximal rectangle* of $\mathbb{K}$. We denote the set of all maximal rectangles by $\mathrm{Rec}(\mathbb{K})$. The set of all formal concepts of $\mathbb{K}$ is denoted by $\mathfrak{B}(\mathbb{K})$ and defines the *concept lattice* $\underline{\mathfrak{B}}(\mathbb{K})$, via the order $(A_1, B_1) \leq (A_2, B_2) :\Longleftrightarrow A_1 \subseteq A_2$. The *complementary context* is defined as $\mathbb{K}^c = (G, M, I^c) := (G, M, (G \times M) - I)$. Furthermore, two special formal concepts of importance are the *object concepts* $\gamma(g) := (\{g\}_I^I, \{g\}^I)$ and *attribute concepts* $\mu(m) = (\{m\}_I, \{m\}_I^I)$.

The *order 2-dimension* of a concept lattice, $\dim_2(\underline{\mathfrak{B}}(\mathbb{K}))$, is the smallest n such that an *order embedding*, that is an order preserving and reflecting map, from $\underline{\mathfrak{B}}(\mathbb{K})$ to the powerset lattice $\underline{2}^X := (2^X, \subseteq)$ of an n-element set X exists. A *Ferrers relation* is a relation $F \subseteq G \times M$ such that $(g, m), (h, n) \in F$ implies $(g, n) \in F$ or $(h, m) \in F$. This is equivalent to $\underline{\mathfrak{B}}(G, M, F)$ being a chain. The *length l* of F is defined as $l(F) = |\mathfrak{B}(G, M, F)| - 1$. For a context $\mathbb{K}$ its *Ferrers 2-dimension*, $\mathrm{fdim}_2(\mathbb{K})$, is the smallest number of Ferrers relations F_t, $t \in T$ with $l(F_t) < 2$, so that $I = \bigcap_{t \in T} F_t$.

The above defined dimensions are equal and are related to the rectangle cover number via the complementary context, that is:

$$\mathrm{rc}(\mathbb{K}) = \mathrm{fdim}_2(\mathbb{K}^c) = \dim_2(\underline{\mathfrak{B}}(\mathbb{K}^c)).$$

For two contexts $\mathbb{K}_1 = (G_1, M_1, I_1)$ and $\mathbb{K}_2 = (G_2, M_2, I_2)$, we use notation from [4] and define the *cardinal product* $\mathbb{K}_1 \hat{\times} \mathbb{K}_2 := (G_1 \times G_2, M_1 \times M_2, I_1 \hat{\times} I_2)$,

$$((g, h), (m, n)) \in I_1 \hat{\times} I_2 :\Longleftrightarrow (g, m) \in I_1 \text{ and } (h, n) \in I_2.$$

3   The Four Optimization Problems on Posets

Let $\mathbb{P} := (P, \leq)$ be a *finite partially ordered set*, *i.e*, the relation $\leq$ is reflexive, antisymmetric and transitive. For any $R \subseteq X \times Y$, the relation R^{op} is defined as $R^{\mathrm{op}} := \{(y, x) \mid (x, y) \in R\}$. With that, the *dual* of $\mathbb{P}$ is defined as:

$$\mathbb{P}^{\mathrm{op}} = (P, \geq) := (P, \leq^{\mathrm{op}}).$$

For $p \in P$, we define the *principal ideal* $(p] := \{x \in P \mid x \leq p\}$ as well as the *principal filter* $[p) := \{x \in P \mid p \leq x\}$.

Finally, a remark about notation. We will express "there exists at most one" with the quantifier $\exists_{\leq 1}$. For symmetry reasons, we will denote "there exists at least one" with the quantifier $\exists_{\geq 1}$.

3.1   Basic Definitions

We define four relations $\mathrm{Cov}_{\mathbb{P}}, \mathrm{Isol}_{\mathbb{P}}, \mathrm{Block}_{\mathbb{P}}, \mathrm{Pack}_{\mathbb{P}} \subseteq 2^P \times 2^P$:

$$(A, B) \in \mathrm{Cov}_{\mathbb{P}} :\Longleftrightarrow \forall (a \in A) \, \exists_{\geq 1}(b \in B) : a \leq b,$$
$$(A, B) \in \mathrm{Isol}_{\mathbb{P}} :\Longleftrightarrow \forall (b \in B) \, \exists_{\leq 1}(a \in A) : a \leq b,$$
$$(A, B) \in \mathrm{Block}_{\mathbb{P}} :\Longleftrightarrow \forall (b \in B) \, \exists_{\geq 1}(a \in A) : a \leq b,$$
$$(A, B) \in \mathrm{Pack}_{\mathbb{P}} :\Longleftrightarrow \forall (a \in A) \, \exists_{\leq 1}(b \in B) : a \leq b.$$

Definition 1. *Let $A, B \subseteq P$. In case that $(A, B) \in \mathrm{Cov}_{\mathbb{P}}$ holds, we say that B covers A, or that A is* covered *by B. Relation $\mathrm{Isol}_{\mathbb{P}}$ is expressed as A is* isolated *w.r.t. B.*

Fig. 1. The left image depicts $\mathrm{Cov}_{\mathbb{P}}$ (every a is at least below one b) and the right one $\mathrm{Isol}_{\mathbb{P}}$ (every b has at most one a below).

B covers A A is isolated w.r.t. to B
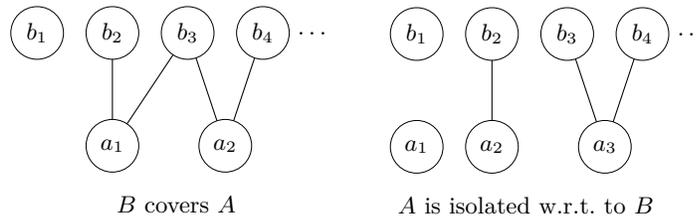
The terminology of a covering is well established through out various branches of mathematics. Presumably, the term isolation is less well known. In [1] is a recent example of its use in matrix theory.

Definition 2. *Let $A, B \subseteq P$. In case that $(A, B) \in \mathrm{Block}_{\mathbb{P}}$ holds, we say that A blocks B, or that B is* blocked *by A. Relation $\mathrm{Pack}_{\mathbb{P}}$ is expressed as B is* packed *w.r.t. A.*

Fig. 2. The left image depicts $\text{Block}_\mathbb{P}$ (every b has at least one a below) and the right one $\text{Pack}_\mathbb{P}$ (every a is at most below one b).

A blocks B B is packed w.r.t. A.
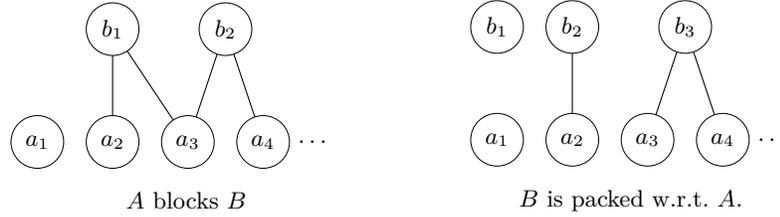
Again, the terminology of a packing is well known. The less well established term blocking is for example used in geometry (see [2,3]).

Via a "twofold dual" the relations can be transformed into each other.

Proposition 1. *It holds that:*

1. $\text{Isol}_\mathbb{P} = (\text{Pack}_{\mathbb{P}^{\text{op}}})^{\text{op}}$ *and* $\text{Pack}_\mathbb{P} = (\text{Isol}_{\mathbb{P}^{\text{op}}})^{\text{op}}$,
2. $\text{Cov}_\mathbb{P} = (\text{Block}_{\mathbb{P}^{\text{op}}})^{\text{op}}$ *and* $\text{Block}_\mathbb{P} = (\text{Cov}_{\mathbb{P}^{\text{op}}})^{\text{op}}$.

Proof.

$$(A, B) \in \text{Isol}_\mathbb{P} \Leftrightarrow \forall (a \in A)\, \exists_{\leq 1}(b \in B) : b \geq a \Leftrightarrow (B, A) \in \text{Pack}_{\mathbb{P}^{\text{op}}}.$$

The other equations can be shown similarly.   □

Next, we define four optimization problems on $\text{Cov}_\mathbb{P}, \text{Isol}_\mathbb{P}, \text{Pack}_\mathbb{P}$ and $\text{Block}_\mathbb{P}$.

Definition 3. *Let* $(A, B) \in \text{Cov}_\mathbb{P}$. *The* cover number *of* (A, B) *is defined as*

$$\text{cov}_\mathbb{P}(A, B) := \min\{\#B_0 \mid B_0 \subseteq B : (A, B_0) \in \text{Cov}_\mathbb{P}\}.$$

We call a subset $B_0 \subseteq B$, *such that* $(A, B_0) \in \text{Cov}_\mathbb{P}$ *and* $\#B_0 = \text{cov}_\mathbb{P}(A, B)$, *a* minimal cover *of* A.

Definition 4. *For* $(A, B) \in \text{Block}_\mathbb{P}$, *the* blocking number *of* (A, B) *is defined as*

$$\text{block}_\mathbb{P}(A, B) := \min\{\#A_0 \mid A_0 \subseteq A : (A_0, B) \in \text{Block}_\mathbb{P}\}.$$

We call a subset $A_0 \subseteq A$, *such that* $(A_0, B) \in \text{Block}_\mathbb{P}$ *and* $\#A_0 = \text{block}_\mathbb{P}(A, B)$, *a* minimal blocking set.

Definition 5. *The* isolation number *and the* packing number *of* (A, B) *are defined as:*

$$\text{isol}_\mathbb{P}(A, B) := \max\{\#A_0 \mid A_0 \subseteq A : (A_0, B) \in \text{Isol}_\mathbb{P}\},$$
$$\text{pack}_\mathbb{P}(A, B) := \max\{\#B_0 \mid B_0 \subseteq B : (A, B_0) \in \text{Pack}_\mathbb{P}\}.$$

We call $A_0 \subseteq A$, *with* $(A_0, B) \in \text{Isol}_\mathbb{P}$ *and* $\#A_0 = \text{isol}_\mathbb{P}(A, B)$, *a* maximal isolated set *w.r.t.* B, *and* $B_0 \subseteq B$, *with* $(A, B_0) \in \text{Pack}_\mathbb{P}$ *and* $\#B_0 = \text{pack}_\mathbb{P}(A, B)$, *a* maximal packing *w.r.t.* A.

Note that, contrary to the cover- and blocking number, the isolation- and packing number can be defined without any condition on (A, B). For example, to find a minimal cover, we have to claim that B covers A. Otherwise there couldn't be a solution to the cover problem. Since it always holds that $(\emptyset, B) \in \mathrm{Isol}_\mathbb{P}$ and $(A, \emptyset) \in \mathrm{Pack}_\mathbb{P}$, a valid isolation- and packing number always exists.

Example 1. *Let $\mathbb{K} = (G, M, I)$ be a formal context. If we choose $\mathbb{P} = 2^{G \times M}$, $A = \{\{(g, m)\} \mid (g, m) \in I\} = \binom{I}{1}$ and $B = \mathrm{Rec}(\mathbb{K})$, we can express the four optimization problems from Section 1 in the abstract poset framework.*

$$\mathrm{rc}(\mathbb{K}) = \mathrm{cov}_\mathbb{P}(A, B),\ \mathrm{ri}(\mathbb{K}) = \mathrm{isol}_\mathbb{P}(A, B),$$

$$\mathrm{rp}(\mathbb{K}) = \mathrm{pack}_\mathbb{P}(A, B),\ \mathrm{rb}(\mathbb{K}) = \mathrm{block}_\mathbb{P}(A, B).$$

Proposition 2. *Let $(A, B) \in \mathrm{Cov}_\mathbb{P}$. It holds that $\mathrm{isol}_\mathbb{P}(A, B) \leq \mathrm{cov}_\mathbb{P}(A, B)$.*

Proof. Let A_0 be a maximal isolated set w.r.t. B, that is A_0 is a maximal subset of A such that $\forall (b \in B) \exists_{\leq 1}(a \in A_0) : a \leq b$. Assuming that B_0 is a minimal cover of A, with $\#B_0 < \#A_0$, there exists $a_1, a_2 \in A_0$, with $a_1 \neq a_2$, $a_1 \leq b$ and $a_2 \leq b$ for some $b \in B_0$. This contradicts A_0 being isolated w.r.t. B. □

Proposition 3. *Let $(A, B) \in \mathrm{Block}_\mathbb{P}$. It holds that $\mathrm{pack}_\mathbb{P}(A, B) \leq \mathrm{block}_\mathbb{P}(A, B)$.*

Proof. Let $\mathrm{pack}_\mathbb{P}(A, B) > \mathrm{block}_\mathbb{P}(A, B)$. From Proposition 1, it follows that $\mathrm{isol}_{\mathbb{P}^\mathrm{op}}(B, A) > \mathrm{cov}_{\mathbb{P}^\mathrm{op}}(B, A)$, which is a contradiction to Proposition 2. □

3.2   Behaviour under Order Preserving/Reversing Maps

In this subsection, we will explore the behaviour of the four optimization problems under poset mappings from $\mathbb{P}$ to $\mathbb{Q}$. Especially, we are interested in maps with the following properties.

Definition 6. *Let $A, B \subseteq \mathbb{P}$. A map $\varphi : \mathbb{P} \to \mathbb{Q}$:*

- *preserves order between A and B :$\Leftrightarrow \forall (a \in A, b \in B) : a \leq b \Rightarrow \varphi(a) \leq \varphi(b)$,*
- *reverses order between A and B :$\Leftrightarrow \forall (a \in A, b \in B) : a \leq b \Rightarrow \varphi(b) \leq \varphi(a)$,*
- *reflects order between A and B :$\Leftrightarrow \forall (a \in A, b \in B) : a \leq b \Leftarrow \varphi(a) \leq \varphi(b)$,*
- *deflects order between A and B :$\Leftrightarrow \forall (a \in A, b \in B) : a \leq b \Leftarrow \varphi(b) \leq \varphi(a)$.*

By means of such maps, it is possible to relate the four relations to each other.

Proposition 4. *For $A, B \subseteq \mathbb{P}$, let $\varphi : \mathbb{P} \to \mathbb{Q}$ be order preserving, and $\psi : \mathbb{P} \to \mathbb{Q}$ order reversing, between A and B. It holds that:*

$$(A, B) \in \mathrm{Cov}_\mathbb{P} \Rightarrow (\varphi[A], \varphi[B]) \in \mathrm{Cov}_\mathbb{Q}\ and\ (\psi[B], \psi[A]) \in \mathrm{Block}_\mathbb{Q}.$$

Similarly, this statement holds if we swap Cov and Block.

Proof. Let $(A, B) \in \mathrm{Cov}_\mathbb{P}$. The statement, for all $\tilde{a} \in \varphi[A]$, there exists at least one $\tilde{b} \in \varphi[B]$ with $\tilde{a} \leq \tilde{b}$, is true, since there exists $a \in A$ and $b \in B$ such that $\varphi(a) = \tilde{a}$ and $\varphi(b) = \tilde{b}$. Hence, $\tilde{a} = \varphi(a) \leq \varphi(b) = \tilde{b}$.

Considering ψ as an order preserving map from $\mathbb{P}$ to $\mathbb{Q}^\mathrm{op}$, it follows that $(\psi[A], \psi[B]) \in \mathrm{Cov}_{\mathbb{Q}^\mathrm{op}}$. Proposition 1 implies that $(\psi[B], \psi[A]) \in \mathrm{Block}_\mathbb{Q}$. □

Proposition 5. *For $A, B \subseteq \mathbb{P}$, let $\varphi : \mathbb{P} \to \mathbb{Q}$ be order reflecting, and $\psi : \mathbb{P} \to \mathbb{Q}$ order deflecting, between A and B. It holds that:*

$$(A, B) \in \mathrm{Isol}_{\mathbb{P}} \Rightarrow (\varphi[A], \varphi[B]) \in \mathrm{Isol}_{\mathbb{Q}} \ and \ (\psi[B], \psi[A]) \in \mathrm{Pack}_{\mathbb{Q}} \,.$$

Similarly, this statement holds if we swap Isol *and* Pack.

Proof. Let $(A, B) \in \mathrm{Isol}_{\mathbb{P}}$. The statement, for all $\tilde{b} \in \varphi[B]$, there exists at most one $\tilde{a} \in \varphi[A]$ with $\tilde{a} \leq \tilde{b}$, is true, since φ reflects order. There can not be any "new edges created by φ" between $\varphi[A]$ and $\varphi[B]$.

Considering ψ as an order reflecting map from $\mathbb{P}$ to $\mathbb{Q}^{\mathrm{op}}$, it follows that $(\psi[A], \psi[B]) \in \mathrm{Isol}_{\mathbb{Q}^{\mathrm{op}}}$. Proposition 1 implies that $(\psi[B], \psi[A]) \in \mathrm{Pack}_{\mathbb{Q}}$. $\square$

Combining Proposition 4 and 5, we get:

Proposition 6. *For $A, B \subseteq \mathbb{P}$, let $\varphi : \mathbb{P} \to \mathbb{Q}$ be order preserving and reflecting, and $\psi : \mathbb{P} \to \mathbb{Q}$ order reversing and deflecting, between A and B.*

$$(A, B) \in \mathrm{Cov}_{\mathbb{P}} \Leftrightarrow (\varphi[A], \varphi[B]) \in \mathrm{Cov}_{\mathbb{Q}} \ and \ (\psi[B], \psi[A]) \in \mathrm{Block}_{\mathbb{Q}},$$
$$(A, B) \in \mathrm{Isol}_{\mathbb{P}} \Leftrightarrow (\varphi[A], \varphi[B]) \in \mathrm{Isol}_{\mathbb{Q}} \ and \ (\psi[B], \psi[A]) \in \mathrm{Pack}_{\mathbb{Q}} \,.$$

Similarly, this statement holds if we swap Cov *and* Block, *or,* Isol *and* Pack.

Proof. The proof either follows or is analogous to the ones of Proposition 4 and 5. Exemplary, we treat $(\varphi[A], \varphi[B]) \in \mathrm{Cov}_{\mathbb{Q}} \Rightarrow (A, B) \in \mathrm{Cov}_{\mathbb{P}}$.
It has to hold that for all $a \in A$ there is at least one $b \in B$ with $a \leq b$. To conclude this from $(\varphi[A], \varphi[B]) \in \mathrm{Cov}_{\mathbb{Q}}$, the map φ has to be order reflecting between A and B. $\square$

Finally, the next theorem describes the behaviour of the four optimization problems under certain poset maps.

Theorem 1. *For $A, B \subseteq \mathbb{P}$, let $\varphi : \mathbb{P} \to \mathbb{Q}$ be order preserving and reflecting, and $\psi : \mathbb{P} \to \mathbb{Q}$ order reversing and deflecting, between A and B.*

For $(A, B) \in \mathrm{Cov}_{\mathbb{P}}$, it holds that:

$$\mathrm{cov}_{\mathbb{P}}(A, B) = \mathrm{cov}_{\mathbb{Q}}(\varphi[A], \varphi[B]) = \mathrm{block}_{\mathbb{Q}}(\psi[B], \psi[A]).$$

Generally, it holds that:

$$\mathrm{isol}_{\mathbb{P}}(A, B) = \mathrm{isol}_{\mathbb{Q}}(\varphi[A], \varphi[B]) = \mathrm{pack}_{\mathbb{Q}}(\psi[B], \psi[A]).$$

Similar equations hold if we swap cov *and* block *(assuming $(A, B) \in \mathrm{Block}_{\mathbb{P}}$), as well as* isol *and* pack.

Proof. We prove that $\mathrm{cov}_{\mathbb{P}}(A, B) = \mathrm{cov}_{\mathbb{Q}}(\varphi[A], \varphi[B])$. The other equations can be shown analogously.

Let $B_0 \subseteq B$ be a minimal cover of A. Since it holds that $(\varphi[A], \varphi[B_0]) \in \mathrm{Cov}_{\mathbb{Q}}$ (Proposition 6), it follows that $\mathrm{cov}_{\mathbb{Q}}(\varphi[A], \varphi[B]) \leq \mathrm{cov}_{\mathbb{P}}(A, B)$. Conversely, let $\tilde{B}_0 \subseteq \varphi[B]$ be a minimal cover of $\varphi[A]$. There exists a subset $B_0 \subseteq B$ such that $\varphi[B_0] = \tilde{B}_0$ and $\#B_0 = \#\tilde{B}_0$. It holds that $(A, B_0) \in \mathrm{Cov}_{\mathbb{P}}$ (Proposition 6) and hence that $\mathrm{cov}_{\mathbb{P}}(A, B) \leq \mathrm{cov}_{\mathbb{Q}}(\varphi[A], \varphi[B])$. $\square$

The first example of a poset map that we will consider is an *order reversing involution* $(\_)^* : \mathbb{P} \to \mathbb{P}$. That is for all $p \in P$ it holds that $p^{**} = p$. Such a map is necessarily a bijection. We will denote the image of $X \subseteq P$ under $(\_)^*$ by X^*.

Theorem 2. *Let* $(\_)^* : \mathbb{P} \to \mathbb{P}$ *be an order reversing involution and* $A, B \subseteq \mathbb{P}$. *It holds that:*

1. $(A, B) \in \mathrm{Cov}_\mathbb{P} \Rightarrow \mathrm{cov}_\mathbb{P}(A, B) = \mathrm{block}_\mathbb{P}(B^*, A^*)$,
2. $\mathrm{isol}_\mathbb{P}(A, B) = \mathrm{pack}_\mathbb{P}(B^*, A^*)$.

Similar equations hold if we swap cov *and* block *(assuming* $(A, B) \in \mathrm{Block}_\mathbb{P}$*), as well as* isol *and* pack*.*

Proof. Let $a \in A$ and $b \in B$. If $a^* \leq b^*$ it follows that $b = b^{**} \leq a^{**} = a$. Hence, $(\_)^*$ is order deflecting (and reversing by definition) between A and B. That means we can apply Theorem 1. $\qquad\square$

Example 2. *Let* $\mathbb{K} = (G, M, I)$ *be a formal context. The complement* $(\_)^c$, *defined in Section 2, is an involution on* $\mathbb{P} = 2^{G \times M}$. *With* $A = \binom{I}{1}$, $B = \mathrm{Rec}(\mathbb{K})$ *and Theorem 2, it follows that the Ferrers 2-dimension can be interpreted as a blocking problem. That is, find the minimal number of maximal rectangle complements (these are 2-step Ferrers relations), such that every coatom* $\{(g, m)\}^c$ *contains at least one. Since the intersection of all coatoms is equal to* I^c, *the intersection of this minimal number of rectangle complements must be too.*

Fig. 3. The complementation in $2^{G \times M}$.
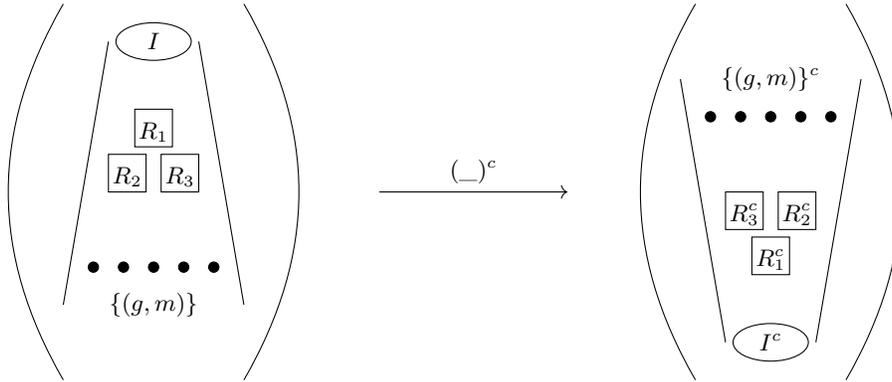
The second example of a poset map that we want to investigate is based on the *powerset representations* of $\mathbb{P}$. These are an order preserving and an order revering map into 2^P.

$$\mathrm{rep}{\downarrow} : \mathbb{P} \to 2^P, \ p \mapsto (p] \qquad \text{and} \qquad \mathrm{rep}{\upharpoonright} : \mathbb{P} \to 2^P, \ p \mapsto [p).$$

For $X \subseteq P$ these maps restrict to 2^X, via

$$\text{rep}{\downarrow}_X : \mathbb{P} \to 2^X,\ p \mapsto (p] \cap X \qquad \text{and} \qquad \text{rep}{\upharpoonright}_X : \mathbb{P} \to 2^X,\ p \mapsto [p) \cap X.$$

Theorem 3. *Let $(A, B) \in \text{Block}_{\mathbb{P}}$. It holds that:*

1. $(A, B) \in \text{Cov}_{\mathbb{P}} \Rightarrow \text{cov}_{\mathbb{P}}(A, B) = \text{cov}_{2^P}(\text{rep}{\downarrow}_A[A], \text{rep}{\downarrow}_A[B])$,
2. $\text{isol}_{\mathbb{P}}(A, B) = \text{isol}_{2^P}(\text{rep}{\downarrow}_A[A], \text{rep}{\downarrow}_A[B])$.

Similar equations hold if we replace cov *with* block *(and* Cov *with Block), as well as* isol *with* pack.

Proof. The requirement $(A, B) \in \text{Block}_{\mathbb{P}}$ is there to assure that for all $b \in B$ there is at least one $a \in A$ such that $a \leq b$. Otherwise there could exist $b \in B$, such that $\text{rep}{\downarrow}_A(b) = (b] \cap A = \emptyset$. If we exclude this case, we can show that the map $\text{rep}{\downarrow}_A$ preserves and reflects order between A and B. Hence, Theorem 1 applies. For $a \in A$ and $b \in B$, we have that:

$$\text{rep}{\downarrow}_A(a) = (a] \cap A = \{x \mid x \in A \wedge x \leq a\},$$
$$\text{rep}{\downarrow}_A(b) = (b] \cap A = \{x \mid x \in A \wedge x \leq b\}.$$

If $a \leq b$, then $(a] \subseteq (b]$ and consequently $(a] \cap A \subseteq (b] \cap A$. If conversely $(a] \cap A \subseteq (b] \cap A$, then a must be an element of $(b]$ which implies that $a \leq b$. $\square$

Example 3. *We consider $\mathbb{K} = (G, M, I)$ and put $\mathbb{P} = 2^{G \times M}$, $A = \binom{I}{1}$ and $B = \text{Rec}(\mathbb{K})$. Since every maximal rectangle contains at least one $\{(g, m)\} \in A$ it holds that $(A, B) \in \text{Block}_{\mathbb{P}}$.*

The effect of applying $\text{rep}{\downarrow}_A$ to A and B is that we "add one more layer of curly brackets". A maximal rectangle $R = \{(g_1, m_1), \ldots, (g_k, m_k)\}$ will be mapped to $\{\{(g_1, m_1)\}, \ldots, \{(g_k, m_k)\}\}$ and every $\{(g, m)\} \in A$ to $\{\{(g, m)\}\}$. Obviously, the four optimization problems will not be affected by this transformation.

Theorem 4. *Let $(A, B) \in \text{Cov}_{\mathbb{P}}$. It holds that:*

1. $\text{cov}_{\mathbb{P}}(A, B) = \text{block}_{2^P}(\text{rep}{\upharpoonright}_B[B], \text{rep}{\upharpoonright}_B[A])$,
2. $\text{isol}_{\mathbb{P}}(A, B) = \text{pack}_{2^P}(\text{rep}{\upharpoonright}_B[B], \text{rep}{\upharpoonright}_B[A])$.

Similar equations hold if we swap cov *and* block *(assuming $(A, B) \in \text{Block}_{\mathbb{P}}$), as well as* isol *and* pack.

Proof. Similar to the proof of Theorem 3. One has to show that $\text{rep}{\upharpoonright}_B$ reverses and deflects order, and can apply Theorem 1. $\square$

Example 4. *Let $\mathbb{K} = (G, M, I)$ be a formal context. Like in Example 3, we put $\mathbb{P} = 2^{G \times M}$, $A = \binom{I}{1}$ and $B = \text{Rec}(\mathbb{K})$. Obviously, $(A, B) \in \text{Cov}_{\mathbb{P}}$. For every pair $(g, m) \in I$, we define the set of all rectangles containing (g, m) as $\text{Rec}(g, m) := \{R \mid R \in \text{Rec}(\mathbb{K}), (g, m) \in R\}$. The image of A and B under $\text{rep}{\upharpoonright}_B$ is then given as:*

$$\text{rep}{\upharpoonright}_B[A] = \{\text{Rec}(g, m) \mid (g, m) \in I\},$$
$$\text{rep}{\upharpoonright}_B[B] = \{\{R\} \mid R \in \text{Rec}(\mathbb{K})\}.$$

Instead of looking on the optimization problems w.r.t. these two sets, we are going to reinterpret them. Every $C \times D \in \mathrm{Rec}(\mathbb{K})$, can be replaced with a formal concept (C, D). This gives us almost $\mathfrak{B}(\mathbb{K})$, but possibly without $(\emptyset, M)$ and $(G, \emptyset)$. Since it holds that

$$(g, m) \in C \times D \iff (C, D) \in [\gamma(g), \mu(m)],$$

the set of all maximal rectangles containing (g, m) can be interpreted as the interval $[\gamma(g), \mu(m)]$ in the concept lattice. This point of view lifts the four optimization problems on formal contexts to the concept lattice $\underline{\mathfrak{B}}(\mathbb{K})$ or more precisely to $2^{\mathfrak{B}(\mathbb{K})}$. Especially, by means of Theorem 4, this provides a meaningfully interpretation to the rectangle isolation- and rectangle blocking number.

ri($\mathbb{K}$): *Find the largest number of disjoint intervals $[\gamma(g), \mu(m)]$ (packing problem).*

rb($\mathbb{K}$): *Find the smallest number of intervals $[\gamma(g), \mu(m)]$ that cover all formal concepts (C, D) with $C, D \neq \emptyset$ (cover problem).*

Since ri($\mathbb{K}$) $\leq$ rc($\mathbb{K}$), another interesting consequence is the fact that the maximal number of disjoint intervals $[\gamma(g), \mu(m)]$ from $\underline{\mathfrak{B}}(\mathbb{K})$ provides a lower bound for the order 2-dimension of $\underline{\mathfrak{B}}(\mathbb{K}^c)$.

Next, we combine Theorem 3 and 4.

Theorem 5. *Let $(A, B) \in \mathrm{Cov}_{\mathbb{P}} \cap \mathrm{Block}_{\mathbb{P}}$.*

Fig. 4. A commutative diagram depicting various poset maps.

1. *The diagram from Figure 4 commutes.*
2. *Defining $\varphi := 2^{(\_)^*} \circ \mathrm{rep}{\restriction}_B$, we get $\mathrm{cov}_{\mathbb{P}}(A, B) = \mathrm{block}_{2^{B^*}}(\varphi[A], \varphi[B])$.*
3. *Defining $\psi := 2^{(\_)^*} \circ \mathrm{rep}{\downarrow}_A$, we get $\mathrm{cov}_{\mathbb{P}}(A, B) = \mathrm{cov}_{2^{A^*}}(\psi[B], \psi[A])$.*
4. *Similar equations hold if we swap* cov *and* block.
5. *Similar equations hold for* isol *and* pack.

Proof. 1. We prove commutativity of the right diagram. Commutativity of the left one can be shown analogously. For $p \in P$, there are two possible images:

$$p \mapsto [p) \cap B \mapsto \{x \mid x \in [p) \cap B\}^* = \{x^* \mid p \leq x \wedge x \in B\} =: S_1,$$
$$p \mapsto p^* \mapsto (p^*] \cap B^* = \{x \mid x \leq p^* \wedge x \in B^*\} =: S_2.$$

If $p \leq x$, then $x^* \leq p^*$ and if $x \in B$ then $x^* \in B^*$. Hence, it holds that $S_1 \subseteq S_2$. For $y \in S_2$, we have to find $\tilde{y} \in P$ with $p \leq \tilde{y}$ and $\tilde{y} \in B$, such that $\tilde{y}^* = y$. Since $(\_)^*$ is an involution $\tilde{y} := y^*$ has this property. Consequently, we conclude that $S_2 \subseteq S_1$.

2. Since, $2^{(\_)^*}$ preserves and reflects order between $\text{rep}\!\restriction_B[B]$ and $\text{rep}\!\restriction_B[A]$, and $\text{rep}\!\restriction_B$ is order reversing and order deflecting between A and B, their composite φ is order reversing and order deflecting between A and B too. Hence, Theorem 1 can be applied.

3. Similar to 2. $\square$

Example 5. *Again, we consider $\mathbb{K} = (G, M, I)$ and put $\mathbb{P} = 2^{G \times M}$, $A = \binom{I}{1}$ and $B = \text{Rec}(\mathbb{K})$. Looking at the diagram from Figure 4, we see that Example 2 took us to the lower center, Example 3 to the upper left corner and Example 4 to the upper right corner. Hence, only the lower corners are left to explore. The diagram from Figure 3 can help to construct the necessary images from A and B under the respective maps. We will only sketch it and begin with the lower right corner from the diagram in Figure 4:*

$$\{(g, m)\} \mapsto \{(g, m)\}^c \mapsto \{R^c \mid R^c \subseteq \{(g, m)\}^c\},$$

$$R \mapsto R^c \mapsto \{R^c\}.$$

Hence, for example the rectangle cover problem translates to the blocking problem of finding the minimal number of 2-step Ferrers relations, such that every collection of 2-step Ferrers relations representing an atom $\{(g, m)\}$ contains at least one of them.

The lower left corner from the diagram in Figure 4 translates to:

$$\{(g, m)\} \mapsto \{(g, m)\}^c \mapsto \{\{(g, m)\}^c\},$$

$$R \mapsto R^c \mapsto \{\{(g, m)\}^c \mid R^c \subseteq \{(g, m)\}^c\}.$$

Hence the rectangle cover problem translates to the cover problem of finding the minimal number of collections of coatoms representing one rectangle, such that every coatom is covered.

It seems that these two points of view don't provide new insights to the four optimization problems on formal contexts.

3.3   Behaviour under Products

The direct product of two posets $\mathbb{P}$ and $\mathbb{Q}$ is defined on the Cartesian product $P \times Q$, together with the product order

$$(a,b) \leq (c,d) :\Longleftrightarrow (a \leq c) \wedge (b \leq d).$$

Theorem 6. *For posets $\mathbb{P}$ and $\mathbb{Q}$, let $(A,B) \in \mathrm{Cov}_\mathbb{P}$ and $(C,D) \in \mathrm{Cov}_\mathbb{Q}$. Then $(A \times C, B \times D) \in \mathrm{Cov}_{\mathbb{P} \times \mathbb{Q}}$. It holds that:*

$$l \leq \mathrm{cov}_{\mathbb{P} \times \mathbb{Q}}(A \times C, B \times D) \leq u,$$

$$u = \mathrm{cov}_\mathbb{P}(A,B)\,\mathrm{cov}_\mathbb{Q}(C,D),$$
$$l = \max(\,\mathrm{isol}_\mathbb{P}(A,B)\,\mathrm{cov}_\mathbb{Q}(C,D), \mathrm{cov}_\mathbb{P}(A,B)\,\mathrm{isol}_\mathbb{Q}(C,D)).$$

Similar equations hold if we replace Cov *by* Block, cov *by* block *and* isol *by* pack.

Proof. Since $(A,B) \in \mathrm{Cov}_\mathbb{P}$ and $(C,D) \in \mathrm{Cov}_\mathbb{Q}$, it follows from the definition of the direct product that $\forall((a,c) \in A \times C)\,\exists_{\geq 1}((b,d) \in B \times D) : (a,c) \leq (b,d)$. Hence $(A \times C, B \times D) \in \mathrm{Cov}_{\mathbb{P} \times \mathbb{Q}}$. For the upper bound, let B_0 be a minimal cover of A, and D_0 a minimal cover of C. It follows that, $(A \times C, B_0 \times D_0) \in \mathrm{Cov}_{\mathbb{P} \times \mathbb{Q}}$.

To prove the lower bound, let $\mathcal{T}$ be a minimal cover of $A \times C$ and A_0 a maximal isolated set w.r.t. B. For every $a \in A_0$, we define $\mathcal{T}_a := \{(b,d) \mid (b,d) \in \mathcal{T}$ and $a \leq b\}$. From the definition of an isolated set, it follows that for different $a, \tilde{a} \in A_0$ the sets $\mathcal{T}_a$ and $\mathcal{T}_{\tilde{a}}$ are disjoint. Furthermore, since $\pi_2[\mathcal{T}_a]$ covers C for all $a \in A_0$, it holds that $\#\mathcal{T}_a \geq \mathrm{cov}_\mathbb{Q}(C,D)$. This leads to:

$$\#\mathcal{T} \geq \#\Big(\biguplus_{a \in A_0} \mathcal{T}_a\Big) \geq \#A_0\;\mathrm{cov}_\mathbb{Q}(C,D) = \mathrm{isol}_\mathbb{P}(A,B)\,\mathrm{cov}_\mathbb{Q}(C,D).$$

The second part of the lower bound can be shown similarly. $\square$

Example 6. *For two formal contexts $\mathbb{K}_1$ and $\mathbb{K}_2$ it was shown in [6] that:*

$$\max(\,\mathrm{ri}(\mathbb{K}_1)\,\mathrm{rc}(\mathbb{K}_2), \mathrm{rc}(\mathbb{K}_1)\,\mathrm{ri}(\mathbb{K}_2)) \leq \mathrm{rc}(\mathbb{K}_1 \,\hat{\times}\, \mathbb{K}_2) \leq \mathrm{rc}(\mathbb{K}_1)\,\mathrm{rc}(\mathbb{K}_2).$$

Now, this result is a consequence of Theorem 6.

Corollary 1. *Let $(A,B) \in \mathrm{Cov}_\mathbb{P}$ and $(C,D) \in \mathrm{Cov}_\mathbb{Q}$. If $\mathrm{cov}_\mathbb{P}(A,B) = \mathrm{isol}_\mathbb{P}(A,B)$ or $\mathrm{cov}_\mathbb{Q}(C,D) = \mathrm{isol}_\mathbb{Q}(C,D)$, then the cover number is multiplicative with respect to the direct product:*

$$\mathrm{cov}_{\mathbb{P} \times \mathbb{Q}}(A \times C, B \times D) = \mathrm{cov}_\mathbb{P}(A,B)\,\mathrm{cov}_\mathbb{Q}(C,D),$$

Example 7. *Let $\mathbb{K}_1$ and $\mathbb{K}_2$ be formal contexts. It was shown in [6] that whenever $\mathrm{rc}(\mathbb{K}_1) = \mathrm{ri}(\mathbb{K}_1)$ or $\mathrm{rc}(\mathbb{K}_2) = \mathrm{ri}(\mathbb{K}_2)$ then $\mathrm{rc}(\mathbb{K}_1 \,\hat{\times}\, \mathbb{K}_2) = \mathrm{rc}(\mathbb{K}_1)\,\mathrm{rc}(\mathbb{K}_2)$. Now, this follows from Corollary 1.*

4   Conclusion

In this paper, we analysed the cover-, isolation, packing- and blocking problem on posets. Especially, we focused on mutual relationships between these problems and showed that they can be transformed into each other via special poset maps. Finally, we presented a result about the cover number of the direct product of posets. This led to a sufficient criterion for the multiplicativity of the cover number w.r.t. the direct product.

With regard to Formal Concept Analysis, this abstract framework generalized previously obtained results about the rectangle cover number of the cardinal product of formal contexts. Furthermore, it was shown that the Ferrers 2-dimension can be seen as a blocking problem and we showed a method to interpret the four optimization problems on formal contexts w.r.t. the concept lattice.

Further research should be done on how to interpret the rectangle packing, isolation and blocking problem on formal contexts. Additionally, through this abstract framework, it might be possible to transfer methods, ideas or results, from other areas of mathematics where these four optimization problems have been treated, to Formal Concept Analysis.

References

1. Beasley, L.B., Song, S.Z.: Upper bounds for the isolation number of a matrix over semirings. Mathematics 7 **7(1)**, 65 (2019)
2. Beule, J.D.: Blocking sets and partial spreads in finite polar spaces. Ph.D. thesis, Universiteit Gent (2004)
3. Blokhuis, A., Brouwer, A.E., Wilbrink, H.A.: Blocking sets in pg(2,q) for small p, and partial spreads in pg(3,7). Advances in Geometry **2003**, 245–253 (2003)
4. Deiters, K., Erné, M.: Sums, products and negations of contexts and complete lattices. Algebra Universalis **60**, 469–496 (2009)
5. Ganter, B., Wille, R.: Formal Concept Analysis: Mathematical Foundations. Springer-Verlag New York, Inc. (1997)
6. Jäkel, C., Schmidt, S.E.: Cover problems, dimensions and the tensor product of complete lattices. Proceedings of the Fourteenth International Conference on Concept Lattices and Their Applications, CLA 2018, Olomouc, Czech Republic (2018)