Recognizing Pseudo-Intents is coNP-complete

Mikhail A. Babin, Sergei O. Kuznetsov

State University Higher School of Economics,
Myasnitskaya 20, 101000 Moscow, Russia
{mikleb@yandex.ru, skuznetsov@hse.ru}

Abstract. The problem of recognizing whether a subset of attributes is a pseudo-intent is shown to be coNP-hard, which together with the previous results means that this problem is coNP-complete. Recognizing an essential intent is shown to be NP-complete and recognizing the lectically largest pseudo-intent is shown to be coNP-hard.

1 Introduction

One of the long-standing complexity problems in FCA is the problem of checking whether a given set of attributes is a pseudo-intent. In [4, 5] it was proved that this problem lies in the class co-NP, however, the question whether the problem is complete in this class was still open. In [6] there was a conjecture that this problem is transhyp-hard [6], which would not mean that this problem is co-NP-complete. In this paper we prove a stronger statement, namely that the problem is coNP-hard, which, together with the result from [4, 5] means that the problem is coNP-complete. This main result has several consequences concerning essential intents and lectically largest pseudo-intent. Recognizing an essential intent is NP-complete and recognizing the lectically largest pseudo-intent is coNP-hard. The rest of the paper is organized as follows: In the second section we introduce the main definitions and give a precise problem statement. In the third section we give a proof of the main result. In the fourth section we discuss the complexity of some related problems, namely that of recognizing essential intents and generating pseudo-intents in the order dual to the lectic one.

2 Definitions

Let G and M be sets, called the set of objects and attributes, respectively. Let I be a relation $I \subseteq G \times M$ between objects and attributes: for $g \in G, m \in M, gIm$ holds iff the object g has the attribute m. The triple $\mathbb{K} = (G, M, I)$ is called a *(formal) context*. If $A \subseteq G, B \subseteq M$ are arbitrary subsets, then the *Galois connection* is given by the following *derivation operators*:

$$A' = \{m \in M \mid gIm \ \forall g \in A\}$$

$$B' = \{g \in G \mid gIm \ \forall m \in B\}$$

The pair (A, B), where $A \subseteq G$, $B \subseteq M$, $A' = B$, and $B' = A$ is called a *(formal) concept (of the context* $\mathbb{K}$*)* with *extent* A and *intent* B (in this case we have also $A'' = A$ and $B'' = B$). The set of attributes B is *implied by the set of attributes* A, or the implication $A \to B$ holds, if all objects from G that have all attributes from the set A also have all attributes from the set B, i.e. $A' \subseteq B'$.

The operation $(\cdot)''$ is a closure operator [1], i.e. it is idempotent ($X'''' = X''$), extensive ($X \subseteq X''$), and monotone ($X \subseteq Y \Rightarrow X'' \subseteq Y''$). Sets $A \subseteq G$, $B \subseteq M$ are called *closed* if $A'' = A$ and $B'' = A$. Obviously, extents and intents are closed sets.

Implications obey the Armstrong rules:

$$\frac{}{A \to A}, \quad \frac{A \to B}{A \cup C \to B}, \quad \frac{A \to B, B \cup C \to D}{A \cup C \to D}.$$

A minimal (in the number of implications) subset of implications, from which all other implications of a context can be deduced by means of the Armstrong rules was characterized in [3]. This subset is called the Duquenne Guigues or stem base in the literature. The premises of the implications of the stem base can be given by pseudo-intents(see e.g.[1]): a set $P \subseteq M$ is a *pseudo-intent* if $P \neq P''$ and $Q'' \subset P$ for every pseudo-intent $Q \subset P$. For a closed set $A \subseteq M$ such that $P \nsubseteq A$ the intersection $A \cap P$ is also closed (see [1]). A set $Q \subseteq M$ is called *quasi-closed* (*quasi-intent*) if for any $R \subseteq Q$ one has $R'' \subseteq Q$ or $R'' = Q''$. For example closed sets are quasi-closed. For a quasi-closed set Q it holds that $(Q \cap C)'' = (Q \cap C)$ for any closed set C such that $Q \nsubseteq C$. Another definition of a pseudo-intent, which we will use in this paper, is very close to that from [3]: a nonclosed set $P \subseteq M$ is a pseudo-intent iff P is quasi-closed and $Q'' \subseteq P$ for any quasi-closed set $Q \subset P$ (see [4, 5]). A set $A \subseteq M$ is called an *essential intent* (*essential-closed subset of attributes*) iff there is a pseudo-intent $P \subseteq M$ such that $P'' = A$.

Let $G = \{g_1, \ldots, g_n\}$ and $M = \{m_1, \ldots, m_n\}$ be sets with same cardinality. Then the context $\mathbb{K} = (G, M, \mathcal{I}_{\neq})$ is called *contranominal scale*, where $\mathcal{I}_{\neq} = G \times M \setminus \{(g_1, m_1), \ldots, (g_n, m_n)\}$. The contranominal scale has the following property, which we will use later: for any $H \subseteq M$ one has $H'' = H$ and $H' = \{g_i \mid m_i \notin H, 1 \leq i \leq n\}$.

3 Recognition of pseudo-intents

Here we discuss the algorithmic complexity of the problem of pseudo-intent recognition.

Problem: Pseudo-intent recognition (PI)
INPUT: A context $\mathbb{K} = (G, M, I)$ and a set $P \subseteq M$.
QUESTION: Is P a pseudo-intent of $\mathbb{K}$?

In order to prove coNP-hardness of PI we consider the most well-known NP-complete problem, namely CNF satisfiability.

Problem: CNF satisfiability (SAT)
INPUT: A boolean CNF formula $f(x_1, \ldots, x_n) = C_1 \wedge \ldots \wedge C_k$
QUESTION: Is f satisfiable?

Consider an arbitrary CNF instance $C_1, \ldots, C_k$ with variables $x_1, \ldots, x_n$, where $C_i = (l_{i1} \vee \ldots \vee l_{in_i})$ $(1 \le i \le k)$ are clauses and $l_{ij} \in \{x_1, \ldots, x_n\} \cup \{\neg x_1, \ldots, \neg x_n\}$ $(1 \le i \le k,\ 1 \le j \le n_i)$ are some variables or their negations, called literals. From this instance we construct a context $\mathbb{K} = (G, M, I)$. Define

$$M = \{p, C_1, \ldots, C_k, x_1, \neg x_1, \ldots, x_n, \neg x_n, e\}$$

$$\begin{aligned} G = \{&g_{x_1}, g_{\neg x_1}, \ldots, g_{x_n}, g_{\neg x_n}, g_{CX}, g_C, g_{l_1}, \ldots, g_{l_n}\} \\ &\cup \{g_{l_i}^{x_j} \mid 1 \le i \le n,\ 1 \le j \le n\} \cup \{g_{l_i}^{\neg x_j} \mid 1 \le i \le n,\ 1 \le j \le n\} \end{aligned}$$

For $1 \le i \le n$ define the set $L_i = \{x_1, \neg x_1, \ldots, x_n, \neg x_n\} \setminus \{x_i, \neg x_i\}$. In addition for $1 \le i \le n$ and $1 \le j \le n$ define the sets $L_i^{x_j} = L_i \setminus \{x_j\}$ and $L_i^{\neg x_j} = L_i \setminus \{\neg x_j\}$.

Now we are ready to define I. The relation I is given by two parts. The first part is

$$I \cap \{g_{x_1}, g_{\neg x_1}, \ldots, g_{x_n}, g_{\neg x_n}\} \times M = \mathcal{C} \cup \mathcal{I}_{\neq}$$

$$\begin{aligned} \mathcal{C} = \{&(g_{x_i}, C_j) \mid x_i \notin C_j,\ 1 \le i \le n,\ 1 \le j \le k\} \\ &\cup \{(g_{\neg x_i}, C_j) \mid \neg x_i \notin C_j,\ 1 \le i \le n,\ 1 \le j \le k\} \end{aligned}$$

$$\begin{aligned} \mathcal{I}_{\neq} = \{&g_{x_1}, g_{\neg x_1}, \ldots, g_{x_n}, g_{\neg x_n}\} \times \{x_1, \neg x_1, \ldots, x_n, \neg x_n\} \\ &\setminus \{(g_{x_1}, x_1), (g_{\neg x_1}, \neg x_1), \ldots, (g_{x_n}, x_n), (g_{\neg x_n}, \neg x_n)\} \end{aligned},$$

hence $C_i' \cap \{g_{x_1}, g_{\neg x_1}, \ldots, g_{x_n}, g_{\neg x_n}\}$ is the set of objects which correspond to literals not included in C_i $(1 \le i \le k)$, and $\mathcal{I}_{\neq}$ is the relation of the contranominal scale. The rest of I is given by the object intents

$$g_{CX}' = M \setminus \{p, e\}$$

$$g_C' = \{p\} \cup \{C_1, \ldots, C_k\}$$

$$g_{l_i}' = \{p\} \cup L_i,\ 1 \le i \le n$$

$$g_{l_i}^{x_j}{}' = \{p\} \cup L_i^{x_j},\ 1 \le i \le n,\ 1 \le j \le n$$

$$g_{l_i}^{\neg x_j}{}' = \{p\} \cup L_i^{\neg x_j},\ 1 \le i \le n,\ 1 \le j \le n$$

Note that there are some objects (e.g. g_{l_1} and $g_{l_1}^{x_1}$) with the same intents, but this does not matter.

For any $A \subseteq \{x_1, \neg x_1, \ldots, x_n, \neg x_n\}$ that satisfies $A \cap \{x_i, \neg x_i\} \neq \emptyset$ for $1 \le i \le n$, we define truth assignment ϕ_A:

$$\phi_A(x_i) = \begin{cases} true, & \text{if } x_i \notin A \text{ and } \neg x_i \in A; \\ false, & \text{if } \neg x_i \notin A \text{ and } x_i \in A; \\ false, & \text{otherwise } (x_i \in A \text{ and } \neg x_i \in A); \end{cases}$$

	p	$C_1\ C_2\ \cdots\ C_k$	$x_1\ \neg x_1\ \cdots\ x_n\ \neg x_n$	e
g_{x_1}				
$g_{\neg x_1}$				
$\vdots$		$\mathcal{C}$	$\mathcal{I}_{\neq}$	
g_{x_n}				
$g_{\neg x_n}$				
g_{CX}		$\times \cdots\cdots \times$	$\times \cdots\cdots\cdots\cdots \times$	
g_C	$\times$	$\times \cdots\cdots \times$		
g_{l_1}	$\times$		L_1	
$g_{l_1}^{x_1}$	$\times$		$L_1^{x_1}$	
$g_{l_1}^{\neg x_1}$	$\times$		$L_1^{\neg x_1}$	
$\vdots$	$\vdots$		$\vdots$	
$g_{l_1}^{x_n}$	$\times$		$L_1^{x_n}$	
$g_{l_1}^{\neg x_n}$	$\times$		$L_1^{\neg x_n}$	
$\vdots$	$\vdots$		$\vdots$	
g_{l_n}	$\times$		L_n	
$g_{l_n}^{x_1}$	$\times$		$L_n^{x_1}$	
$g_{l_n}^{\neg x_1}$	$\times$		$L_n^{\neg x_1}$	
$\vdots$	$\vdots$		$\vdots$	
$g_{l_n}^{x_n}$	$\times$		$L_n^{x_n}$	
$g_{l_n}^{\neg x_n}$	$\times$		$L_n^{\neg x_n}$	

Table 1. Context $\mathbb{K}$.

In the case $x_i \notin A$ and $\neg x_i \notin A$ for some $1 \leq i \leq n$, ϕ_A is undefined. Note that for $A \subseteq \{x_1, \neg x_1, \ldots, x_n, \neg x_n\}$ the truth assignment ϕ_A is (correctly) defined iff $A \nsubseteq L_i$ for every $1 \leq i \leq n$.

Symmetrically for a truth assignment ϕ define the set $A_\phi = \{\neg x_i \mid \phi(x_i) = true\} \cup \{x_i \mid \phi(x_i) = false\}$.

Before proving coNP-hardness of PI we prove some auxiliary statements. The following lemma is crucial for the reduction from SAT to the complement of PI.

Lemma 1 *If a subset $A \subseteq \{x_1, \neg x_1, \ldots, x_n, \neg x_n\}$ is closed and $A \nsubseteq g'_{l_i}$ for any $1 \leq i \leq n$ then ϕ_A is defined and ϕ_A satisfies f i.e $f(\phi_A) = true$. Conversely, if a truth assignment ϕ satisfies f, then A_ϕ is closed and $A_\phi \nsubseteq g'_{l_i}$ for every $1 \leq i \leq n$.*

Proof. Let $A \subseteq \{x_1, \neg x_1, \ldots, x_n, \neg x_n\}$ and A is not a subset of any g'_{l_i} ($1 \leq i \leq n$), then $A \nsubseteq L_i$ for any $1 \leq i \leq n$ and hence (by definition of ϕ_A) ϕ_A is defined. Since $\mathcal{I}_{\neq}$ is the relation of contranominal scale and any intent can be expressed as the intersection of object intents, we have $A' = \{g_{x_i} \mid x_i \notin A\} \cup \{g_{\neg x_i} \mid \neg x_i \notin A\} \cup B$, where $B \subseteq G - \{g_{x_1}, g_{\neg x_1}, \ldots, g_{x_n}, g_{\neg x_n}\}$. Since $A \nsubseteq L_i$ for any $1 \leq i \leq n$ we also have $A \nsubseteq L_i^{x_j}$ and $A \nsubseteq L_i^{\neg x_j}$ for every $1 \leq i \leq n$ and $1 \leq j \leq n$. Thus $B = \{g_{CX}\}$.

Suppose $A'' = A$. Then $A \cap \{C_1, \ldots, C_k\} = \emptyset$ and hence for every $1 \leq i \leq k$ there is some $g \in A'$ that $C_i \notin g'$. Since $C_i \in g'_{CX}$ and $A' = \{g_{x_i} \mid x_i \notin A\} \cup \{g_{\neg x_i} \mid \neg x_i \notin A\} \cup \{g_{CX}\}$ the latter means that $g \in \{g_{x_i} \mid x_i \notin A\} \cup \{g_{\neg x_i} \mid \neg x_i \notin A\}$. Then, by definition of the relation $\mathcal{C}$, there is a literal $x_j \notin A$ or $\neg x_j \notin A$ that belongs to C_i. Thus ϕ_A satisfies C_i for every $1 \leq i \leq k$.

Now let ϕ be a truth assignment and $f(\phi) = true$. Obviously, $A_\phi \nsubseteq g'_{l_i}$ for every $1 \leq i \leq n$ (by definition of A_ϕ). Then $A'_\phi = \{g_{x_i} \mid x_i \notin A_\phi\} \cup \{g_{\neg x_i} \mid \neg x_i \notin A_\phi\} \cup \{g_{CX}\}$. Note that $A''_\phi \cap \{x_1, \neg x_1, \ldots, x_n, \neg x_n\} = A_\phi \cap \{x_1, \neg x_1, \ldots, x_n, \neg x_n\}$ and $A_\phi \subseteq g'_{CX}$. Hence A_ϕ is closed iff $A_\phi \cap \{C_1, \ldots, C_k\} = \emptyset$. Assume that $C_i \in A_\phi \cap \{C_1, \ldots, C_k\}$ for some $1 \leq i \leq k$. This means that $C_i \in g'_{x_j}$ and $C_i \in g'_{\neg x_r}$ for every $x_j \notin A_\phi$ and $\neg x_r \notin A_\phi$. But then by definition of the relation $\mathcal{C}$ the clause C_i is not satisfied by ϕ. □

Proposition 2 *For any $1 \leq i \leq n$ if $A \subseteq g'_{l_i}$ then A is closed.*

Proof. Let $A \subseteq g'_{l_i}$ and $p \in A$. Then $A'' = \bigcap_{x_j \notin A} {g_{l_i}^{x_j}}' \cap \bigcap_{\neg x_j \notin A} {g_{l_i}^{\neg x_j}}' = A$. In the case $p \notin A$ we can express A'' as $A'' = (A \cup \{p\})'' \cap g'_{CX} = A$. □

Now we are ready to prove coNP-hardness of PI.

Theorem 3 *PI is coNP-hard.*

Proof. We reduce CNF to the complement of PI. Given a CNF instance $f = C_1 \wedge \ldots \wedge C_k$, we construct a context $\mathbb{K}$ like that described above (see Table 1). We take $P = M \setminus \{e\}$ as a set for deciding whether it a pseudo-intent. Hence the corresponding PI instance is ($\mathbb{K}$, P) and we prove that f is satisfiable if and only if P is not a pseudo-intent of $\mathbb{K}$. Without loss of generality we will assume that for every $1 \le i \le n$ the clause $x_i \vee \neg x_i$ is included in f (it does not affect satisfiability).

($\Rightarrow$) Let f be satisfiable and let ϕ be the truth assignment that satisfies $f(\phi) = true$. Consider the set $Q = \{p\} \cup A_\phi$. As we will see later Q is a pseudo-intent, $Q \subset P$ and $Q'' = M \nsubseteq P$, and hence P is not a pseudo-intent. First let us check that $Q'' = M$. Since $p \in Q$ we should test only that $Q \nsubseteq g'$, where $g \in \{g_C, g_{l_1}, \dots, g_{l_n}\} \cup \{g_{l_i}^{x_j} \mid 1 \le i \le n, 1 \le j \le n\} \cup \{g_{l_i}^{\neg x_j} \mid 1 \le i \le n, 1 \le j \le n\}$. Clearly $Q \nsubseteq g'_C$ because A_ϕ is not empty. By Lemma 1 for any $1 \le i \le n$, $A_\phi \nsubseteq g'_{l_i}$, therefore $Q \nsubseteq g_{l_i}$. Hence $Q \nsubseteq {g_{l_i}^{x_j}}'$ and $Q \nsubseteq {g_{l_i}^{\neg x_j}}'$ ($1 \le i \le n$, $1 \le j \le n$). In order to prove that Q is a pseudo-intent we show that any proper subset of Q is closed. Consider an arbitrary set $A \subset Q$. If $p \in A$ then (since $A \neq Q$) there is a literal $l \in \{x_1, \neg x_1, \dots, x_n, \neg x_n\}$ such that $l \in Q$ and $l \notin A$. Thus by proposition 2 the subset A is closed. Now let $p \notin A$ then if $A = Q \setminus \{p\} = A_\phi$ by lemma 1 the subset A is closed. If $A \neq Q \setminus \{p\}$ then $A \subset A_\phi$ and by proposition 2 the subset A is closed.

($\Leftarrow$) Now let a pseudo-intent Q be a proper subset of P (i.e. $Q \subset P$) and $Q'' \nsubseteq P$. Then Q is not a subset of any object intent of $\mathbb{K}$. Together with the fact of quasi-closedness of Q this implies that $Q \cap g'$ is closed for any $g \in G$. Note that $p \in Q$ since otherwise $Q \subseteq g'_{CX}$. Consider $Q \cap g'_C$. Since $Q \cap g'_C$ is closed and $p \in Q \cap g'_C$, there are only two possibilities: $Q \cap g'_C = p$ or $Q \cap g'_C = g'_C$. Assume $Q \cap g'_C = g'_C$. Then $Q = g'_C \cup B$, where $B \subset \{x_1, \neg x_1, \dots, x_n, \neg x_n\}$ and $B \neq \emptyset$ (because $Q \neq P$ and $Q \neq g'_C$). Consider $Q \cap g'_{CX} = \{C_1, \dots, C_k\} \cup B$. This set must be closed by quasi-closedness of Q. Note that $\{C_1, \dots, C_k\} \cup B \nsubseteq g'_{l_i}$, for any $1 \le i \le n$ and $\{C_1, \dots, C_k\} \cup B \nsubseteq g'_C$ (since $B \neq \emptyset$). Thus $(Q \cap g'_{CX})' \subseteq \{g_{x_1}, g_{\neg x_1}, \dots, g_{x_n}, g_{\neg x_n}\}$. Since $(Q \cap g'_{CX})' \neq \emptyset$ there is a literal $l \in \{x_1, \neg x_1, \dots, x_n, \neg x_n\}$ such that $g_l \in (Q \cap g'_{CX})'$. Then, by definition of g'_l and the fact that some clause C_i contains the literal l we get that $C_i \notin Q \cap g'_{CX}$. Thus $Q \cap g'_C = p$ and $Q \setminus \{p\} = Q \cap g'_{CX} \subseteq \{x_1, \neg x_1, \dots, x_n, \neg x_n\}$. Moreover, $Q \nsubseteq g'_{l_i}$ for every $1 \le i \le n$, hence $\phi = \phi_{Q \setminus \{p\}}$ is (correctly) defined. Since $Q \setminus \{p\}$ is closed by lemma 1, the truth assignment ϕ satisfies f. $\square$

In [4] it was shown that $PI \in$ coNP hence we obtain

Corollary 1. PI is coNP-complete.

4 Recognizing essential intents and lectically largest pseudo-intents

An important problem related to recognizing pseudo-intents is deciding whether a given set is the lectically largest pseudo-intent.

Let $M = \{m_1, \dots, m_n\}$ be a finite set with linear order on it ($m_1 < \dots < m_n$). For sets $A \subseteq M$ and $B \subseteq M$ we say that A *lectically smaller* than B ($A < B$, B is lectically larger than A) if $\exists m_i \in B \setminus A : A \cap \{m_j \in M \mid j < i\} = B \cap \{m_j \in M \mid j < i\}$. It is not hard to see that the lectic order is a linear order on the subsets of M.

Problem: The lectically largest pseudo-intent (LLPI)
INPUT: A context $\mathbb{K} = (G, M, I)$ with linear order on M and a set $P \subseteq M$.
QUESTION: Is P the lectically largest pseudo-intent of $\mathbb{K}$?

Proposition 4 *LLPI is coNP-hard.*

Proof. We reduce SAT to the complement of LLPI as in the proof of Theorem 3 . The linear order on M is: $p < C_1 < \ldots < C_k < x_1 < \neg x_1 < \ldots < x_n < \neg x_n < e$. Since $P = M \setminus \{e\}$ and M is closed, P is the lectically largest pseudo-intent iff P is a pseudo-intent. □

Thus it is impossible to find the lectically largest pseudo-intent in polynomial time unless $P = NP$.

In [8] it was shown that pseudo-intents cannot be enumerated with polynomial delay in the lectic order (unless $P = NP$). Proposition 4 shows that this also cannot be done in the dual order, i.e., the following corollary holds.

Corollary. Pseudo-intents cannot be generated with polynomial delay in the order dual to the lectic one unless $P = NP$.

Another problem related to the problem of recognizing pseudo-intents is that of recognizing essential intents.

Problem: Essential intents recognition (EI)
INPUT: A context $\mathbb{K} = (G, M, I)$ and a set $A \subseteq M$.
QUESTION: Is A an essential intent of $\mathbb{K}$?

Proposition 5 *EI is NP-complete.*

Proof. 1. NP-Hardness. We reduce SAT to EI, in the same way as in the reduction from SAT, to the complement of PI. Let us construct the context $\mathbb{K}_2 = (G, M \setminus \{e\}, I)$, where G, M and I are the sets of objects, attributes and the relation of context $\mathbb{K}$ from the proof of Theorem 3 (see Table 1). Obviously, $M \setminus \{e\}$ is an essential intent of $\mathbb{K}_2$ iff $M \setminus \{e\}$ is not a pseudo-intent of $\mathbb{K}$.

2. Membership in NP. The set A is an essential intent of the context $\mathbb{K} = (G, M, I)$ iff there is a pseudo-intent $P \subseteq M$ such that $P'' = A$. Since a pseudo-intent is an inclusion-minimal quasi-closed set with the same closure (e.g. see [4]), a set A is an essential intent iff there is quasi-closed set $Q \subseteq M$ such that $Q'' = A$. Quasi-closedness can be tested in polynomial time (see [4]). Hence a nondeterministic guess for checking essential-intent A can be a quasi-closed set Q such that $Q'' = A$. □.

Conclusion

A long-standing complexity problem about the complexity of recognizing a pseudo-intent was solved. This problem was shown to be coNP-complete. This main result has several important consequences concerning essential intents and the lectically largest pseudo-intent. Recognizing an essential intent was shown to be NP-complete and recognizing the lectically largest pseudo-intent was shown to be coNP-hard. The latter fact means that pseudo-intents cannot be generated with polynomial delay in the order dual to the lectic one unless $P = NP$. Whether pseudo-intents cannot be generated with polynomial delay (unless $P = NP$) in arbitrary order still remains an important open problem.

References

1. B. Ganter and R. Wille, *Formal Concept Analysis: Mathematical Foundations*; Springer, Berlin (1999).
2. M. Garey and D. Johnson, *Computers and Intractability: A Guide to the Theory of NP-Completeness*; Freeman, San Francisco (1979).
3. J. L. Guigues and V. Duquenne, Familles minimales d'implications informatives resultant d'un tableau de données binaries, *Mathématiques, Informatique et Sciences Humaines*; 95:5-18, (1986).
4. S. O. Kuznetsov and S. A. Obiedkov, Counting pseudo-intents and #P-completeness. In R. Missaoui and J. Schmid, Eds, ICFCA 2006,*Lecture Notes in Computer Science*, vol. 3874, 306-308, Springer-Verlag, (2006).
5. S. O. Kuznetsov and S. A. Obiedkov, Some decision and counting problems of the duquenne-guigues basis of implications, *Discrete Applied Mathematics*, 156(11):1994-2003, (2008).
6. B. Sertkaya, Towards the complexity of recognizing pseudo-intents, In F. Dau and S. Rudolph, Eds, ICCS 2009, *Lecture Notes in Computer Science*, vol. 5662, 284–292, Springer-Verlag, (2009)
7. B. Sertkaya, Some computational problems related to pseudo-intents, In S. Ferre and S. Rudolph, Eds, ICFCA 2009, *Lecture Notes in Computer Science*, vol. 5662, 284–292, Springer-Verlag, (2009)
8. F. Distel, Hardness of enumerating pseudo-intents in the lectic order. In L. Kwuida and B. Sertkaya, Eds, ICFCA 2010, *Lecture Notes in Artificial Intelligence*, vol. 5986, 124–137, Springer-Verlag, (2010)