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Linear Equations over Commutative Rings and Determinantal Ideals

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In this paper, we give a characterization of the class of rings which have the property that the existence of a solution of any system of linear equations is equivalent to the equality of the determinantal ideals of the coefficient matrix and the augmented matrix of the system. When one only considers domains, this characterization is equivalent to that given in [5]; however, the techniques used by the authors of this quoted paper are different from those we use below. Our main technique is the fact that the existence of a solution for a system of equations becomes a local property.

I. Basic Results

Thoughout this paper, R will denote a commutative ring with unit. We shall consider a system of linear equations with coefficients in R given by

$$(S)\colon A\underline{x} = \underline{b},$$

where $A = (a_{ij})$ is a $(m \times n)$-matrix with entries in R and $\underline{x} = (x_1,..., x_n)'$ and $\underline{b} = (b_1,..., b_m)^t$ are column vectors with coordinates in R. We shall denote by $(A \mid \underline{b})$ the augmented matrix of the system; this is, the $m \times (n+1)$-matrix obtained from A by adding the column matrix $\underline{b}$.

For any prime ideal $\mathfrak{p} \subset R$, the system of equations over $R_{\mathfrak{p}}$ obtained from (S) by replacing each coefficient by its image in $R_{\mathfrak{p}}$ via the homomorphism $R \to R_{\mathfrak{p}}$ will be denoted by $(S_{\mathfrak{p}})$. Thus, $(S_{\mathfrak{p}})$ is given by

$$(S_{\mathfrak{p}})\colon \quad A_{\mathfrak{p}}\underline{x} = \underline{b}_{\mathfrak{p}},$$

where $A_{\mathfrak{p}} = (a_{ij}/1)$ and $\underline{b}_{\mathfrak{p}} = (b_1/1,..., b_m/1)^t$.

The following result shows that the existence of a solution for (S) is a local property.

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Proposition 1. *The following statements are equivalent:*

(i) *The system* (S) *has a solution (in* R*).*

(ii) *For every prime ideal* $\mathfrak{p}$ *in* R, *the system* $(S_{\mathfrak{p}})$ *has a solution (in* $R_{\mathfrak{p}}$*).*

(iii) *For every maximal ideal* $\mathfrak{m}$ *in* R, *the system* $(S_{\mathfrak{m}})$ *has a solution (in* $R_{\mathfrak{m}}$*).*

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) is evident. We shall see that (iii) $\Rightarrow$ (i). Assume that for every maximal $\mathfrak{m}$ in R the system $(S_{\mathfrak{m}})$ has a solution given by

$$x_j(\mathfrak{m}) = \frac{\alpha_j(\mathfrak{m})}{s(\mathfrak{m})}, \qquad 1 \leqslant j \leqslant n.$$

Note that there is no loss of generality in the assumption that $x_1(\mathfrak{m}), \ldots, x_n(\mathfrak{m})$ have the same denominator $s(\mathfrak{m}) \notin \mathfrak{m}$.

Then for every $\mathfrak{m}$ one has

$$\sum_{j=1}^{n} a_{ij} \frac{\alpha_j(\mathfrak{m})}{s(\mathfrak{m})} = \frac{b_i}{1}, \qquad 1 \leqslant i \leqslant m.$$

So there exists an element $t_i(\mathfrak{m}) \notin \mathfrak{m}$ such that

$$t_i(\mathfrak{m}) \left(\sum_{j=1}^{n} a_{ij} \alpha_j(\mathfrak{m}) \right) = t_i(\mathfrak{m}) \cdot s(\mathfrak{m}) \cdot b_i, \qquad 1 \leqslant i \leqslant m;$$

whence setting $t(\mathfrak{m}) = \prod_{i=1}^{m} t_i(\mathfrak{m})$ one obtains

$$t(\mathfrak{m}) \left(\sum_{j=1}^{m} a_{ij} \alpha_j(\mathfrak{m}) \right) = t(\mathfrak{m}) \cdot s(\mathfrak{m}) \cdot b_i, \qquad 1 \leqslant i \leqslant m.$$

Now, since $t(\mathfrak{m}) \cdot s(\mathfrak{m}) \notin \mathfrak{m}$, the ideal generated by the elements $t(\mathfrak{m}) \cdot s(\mathfrak{m})$, when $\mathfrak{m}$ ranges over the set of maximal ideals of R, is not contained in any $\mathfrak{m}$ and so it is R. It follows that there exist finitely many maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_\rho$ and elements $\lambda_1, \ldots, \lambda_\rho$ in R such that

$$1 = \sum_{k=1}^{\rho} \lambda_k t(\mathfrak{m}_k) \cdot s(\mathfrak{m}_k).$$

Finally, the elements $x_1, \ldots, x_n$ given by

$$x_j = \sum_{k=1}^{\rho} \lambda_k t(\mathfrak{m}_k) \, \alpha_j(\mathfrak{m}_k), \qquad 1 \leqslant j \leqslant n,$$

are a solution of the system (S).

For each integer $i>0$, the ith order determinantal ideal of the matrix C, i.e., the ideal of R generated by all the $(i\times i)$-minors of the matrix C, will be denoted by $\mathcal{U}_i(C)$. For $i=0$ set $\mathcal{U}_0(C)=R$.

In the sequel we be concerned only with the matrices A, $(A\,|\,\underline{b})$, $A_{\not p}$ involved in a system of linear equations as above. The following properties are immediate consequences of the definitions:

(i) $\mathcal{U}_0(A)\supseteq\mathcal{U}_1(A)\supseteq\cdots\supseteq\mathcal{U}_i(A)\supseteq\cdots$.

(ii) If $\not p$ is a prime ideal in R then for any i,

$$\mathcal{U}_i(A_{\not p})=\mathcal{U}_i(A)\cdot R_{\not p}.$$

(iii) If (S) has a solution then for any i, $\mathcal{U}_i(A)=\mathcal{U}_i(A\,|\,\underline{b})$.

PROPOSITION 2. *Let R be a local ring and (S): $A\underline{x}=\underline{b}$ a system of linear equations over R. The system (S) has a solution in R if the following statements hold for every $i\geqslant 0$:*

(i) $\mathcal{U}_i(A)=\mathcal{U}_i(A\,|\,\underline{b})$.

(ii) *Either $\mathcal{U}_i(A)=(0)$ or $\mathcal{U}_i(A)$ is an ideal generated by a non-zerodivisor of R.*

Proof. Let r be the integer such that $\mathcal{U}_r(A)\neq(0)$ and $\mathcal{U}_{r+1}(A)=(0)$. Since $\mathcal{U}_r(A)$ is generated by the $(r\times r)$-minors of A and by the hypothesis (ii) it is a principal ideal generated by a non-zerodivisor in the local ring R, it follows from Nakayama's lemma that $\mathcal{U}_r(A)$ is in fact generated by an $(r\times r)$-minor. Without loss of generality, one can assume that this generator is the determinant λ of the matrix

$$A'=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r}\\ a_{21} & a_{22} & & a_{2r}\\ \vdots & & \ddots & \\ a_{r1} & a_{r2} & \cdots & a_{rr}\end{bmatrix}.$$

Now consider the system (S') given by

$$\sum_{j=1}^{n} a_{ij}x_j=b_i,\qquad 1\leqslant i\leqslant r.$$

By multiplying on the left the matrix of (S') by the cofactor matrix $(A_{k,i})$ of A' one obtains the system given by

$$\lambda x_i+\sum_{t=1}^{n-r}\left(\sum_{k=1}^{r}A_{ki}a_{k,r+t}\right)x_{r+t}=\sum_{k=1}^{r}A_{ki}b_k,\ 1\leqslant i\leqslant r, \tag{1}$$

where A_{ki} is the cofactor of the entry a_{ki} of A'. Equations (1) can be also written in the form

$$\lambda x_i + \sum_{t=1}^{n-r} \begin{vmatrix} a_{11} & \cdots & a_{1,i-1} & a_{1,r+t} & a_{1,i+1} & \cdots & a_{1r} \\ \vdots & & & & & & \\ a_{r1} & \cdots & a_{r,i-1} & a_{r,r+t} & a_{r,i+1} & \cdots & a_{rr} \end{vmatrix} x_{r+t}$$

$$= \begin{vmatrix} a_{11} & \cdots & a_{1,i-1} & b_1 & a_{1,i+1} & \cdots & a_{1r} \\ \vdots & & & & & & \\ a_{r1} & \cdots & a_{r,i-1} & b_r & a_{r,i+1} & \cdots & a_{rr} \end{vmatrix} .$$

Since λ generates the ideal $\mathscr{U}_r(A) = \mathscr{U}_r(A \mid \underline{b})$, it follows from this last expresion that the system (S') has a solution.

To prove the proposition, is suffices to check that the solutions of (S') are also solutions of (S). For this, we shall show that for $1 \leqslant s \leqslant n-r$ the solutions of (S') satisfy the equation

$$a_{r+s,1} x_1 + \cdots + a_{r+s,n} x_n = b_{r+s},$$

or equivalently, since λ is a non-zerodivisor, they satisfy the equation

$$\sum_{j=1}^{n} \lambda a_{r+s,j} x_j = \lambda b_{r+s}.$$

Substituting λx_i by its value given by Eq. (1) the solutions of (S') satisfy

$$\sum_{j=1}^{n} \lambda a_{r+s,j} x_j = \sum_{j=1}^{r} a_{r+s,j} \left(- \sum_{t=1}^{n-r} \left(\sum_{k=1}^{r} A_{kj} a_{k,r+t} \right) x_{r+t} + \sum_{k=1}^{r} A_{kj} b_k \right)$$
$$+ \sum_{t=1}^{n-r} \lambda a_{r+s,r+t} x_{r+t}$$
$$= \sum_{t=1}^{n-r} \left(\sum_{j=1}^{r} a_{r+s,j} \left(- \sum_{k=1}^{r} A_{kj} a_{k,r+t} \right) + \lambda a_{r+s,r+t} \right) x_{r+t}$$
$$+ \sum_{j=1}^{r} a_{r+s,j} \left(\sum_{k=1}^{r} A_{kj} b_k \right).$$

For $1 \leqslant t \leqslant n-r$ in this expresion the coefficient of x_{r+t} is zero as it coincides with the determinant of the $(r+1) \times (r+1)$-submatrix of A obtained by adding to A' the appropriate parts of the $(r+t)$ column and the $(r+s)$ row of A. So

$$\sum_{j=1}^{n} \lambda a_{r+s,j} x_j = \sum_{j=1}^{r} a_{r+s,j} \left(\sum_{k=1}^{r} A_{kj} b_k \right).$$

On the other hand, since $\mathscr{U}_{r+1}(A \mid \underset{\sim}{b}) = (0)$, the determinant of the $(r+1) \times (r+1)$ submatrix of $(A \mid \underset{\sim}{b})$ obtained by adding to A' the appropriate parts of the $(r+s)$-row and the column $\underset{\sim}{b}$ is zero, one as

$$0 = \lambda b_{r+s} - \sum_{j=1}^{r} a_{r+s,j} \left(\sum_{k=1}^{r} A_{k,j} b_k \right).$$

By comparing the two last expresions, one concludes that the solutions of (S') are also solutions of (S) as desired.

Note that, in the above proof of the proposition, the hypothesis on R to be a local ring is used in the fact that if $a_1, \ldots, a_n$ is a set of generators of a principal ideal I of a local ring, then I is generated by some a_i. However, the result can be generalized to the nonlocal case by using the concept of flat ideal of a ring. Recall that a finitely generated ideal I of R is flat if and only if for any prime ideal $\not{p}$ of R the ideal $I_{\not{p}}$ of $R_{\not{p}}$ is either (0) or generated by a non-zerodivisor.

COROLLARY 3. *Let (S): $A\underset{\sim}{x} = \underset{\sim}{b}$ a system of linear equations over the ring R. The system (S) has a solution in R if the following statements hold for every $i \geqslant 0$:*

(i) $\mathscr{U}_i(A) = \mathscr{U}_i(A \mid \underset{\sim}{b})$.

(ii) *$\mathscr{U}_i(A)$ is a flat ideal of R.*

Proof. By the property (ii) of the determinantal ideals, the system $(S_{\not{p}})$ satisfies the hipothesis of the Proposition 2, so it has a solution. Now by Proposition 1, (S) has a solution in R.

2. THE CHARACTERIZATION THEOREM

DEFINITION 4. A ring R is said to be a Prüfer ring if every finitely generated ideal is flat.

When R is a domain, there exist several characterizations for the Prüfer condition [4, p. 559]. One of them is that R is a Prüfer domain if and only if the localization of R at every prime ideal is a valuation ring. We shall note that when R is not a domain, several definitions of Prüfer rings are used in the literature which are not equivalent among them, and in particular not equivalent to the above one [3, p. 27].

We state below some properties of Prüfer rings which will be used in the sequel, and whose proofs follow from definitions and the quoted references:

(i) R is a Prüfer local ring if and only if R is a valuation ring. [4, p. 137, 559; 7, p. 96].

(ii) R is a Prüfer ring if and only if for every prime ideal $\mathfrak{p}$ of R the ring $R_{\mathfrak{p}}$ is a Prüfer local ring (i.e., R is a valuation ring) [1, p. 41].

(iii) If $\{R_i\}_{1 \leqslant i \leqslant h}$ are Prüfer rings then $\prod_{i=1}^{h} R_i$ is also a Prüfer ring. In particular if n is a square-free integer, then Z/nZ is a Prüfer ring. [2, p. 142].

The following lemma will be used in the main result in this paper stated below.

LEMMA 5. *If R is a local domain which is not a valuation ring, then there exist finitely generated ideals $\mathfrak{a}'$, $\mathfrak{b}'$, $\mathfrak{c}'$ with $\mathfrak{a}' \neq 0$, $\mathfrak{b}' \not\subset \mathfrak{c}'$ such that $\mathfrak{a}' \cdot \mathfrak{b}' \subseteq \mathfrak{a}' \cdot \mathfrak{c}'$.*

Proof. If the domain R is not a valuation ring then it follows from [4, p. 559] that either R is not integrally closed or there exist finitely generated fractional ideals $\mathfrak{a}''$, $\mathfrak{b}''$, $\mathfrak{c}''$ with $\mathfrak{b}'' \neq \mathfrak{c}''$ and $\mathfrak{a}'' \cdot \mathfrak{b}'' = \mathfrak{a}'' \cdot \mathfrak{c}''$.

If R is not integrally closed then by [4, p. 546] there exists a finitely generated ideal $\mathfrak{a}'$ of R such that $(\mathfrak{a}' : \mathfrak{a}') \neq R$. So there exist $x, y \in R$ with $x \notin (y)$ such that $(x/y)\, \mathfrak{a}' \subseteq \mathfrak{a}'$. Hence $\mathfrak{a}'.(x) \subseteq \mathfrak{a}' \cdot (y)$.

If there exist finitely generated fractional ideals $\mathfrak{a}''$, $\mathfrak{b}''$, $\mathfrak{c}''$ with $b'' \neq \mathfrak{c}''$ and $\mathfrak{a}'' \cdot \mathfrak{b}'' = \mathfrak{a}'' \cdot \mathfrak{c}''$, then take $d \in R$, $d \neq 0$, such that $d\mathfrak{a}'' = \mathfrak{a}'$, $d\mathfrak{b}'' = \mathfrak{b}'$ and $d\mathfrak{c}'' = \mathfrak{c}'$ are ideals of R. It is clear that the ideals $\mathfrak{a}'$, $\mathfrak{b}'$, $\mathfrak{c}'$ are finitely generated and satisfy the requirements in the lemma.

THEOREM 6. *For a commutative ring with unit R the following statements are equivalent.*

(i) *A system of linear equations over R, (S): $A\underline{x} = \underline{b}$, has a solution in R if and only if for all i one has*

$$\mathcal{U}_i(A) = \mathcal{U}_i(A \mid \underline{b}).$$

(ii) *R is a Prüfer ring.*

Proof. (ii) $\Rightarrow$ (i) Let (S): $A\underline{x} = \underline{b}$ a system of linear equations over R. If (S) has a solution in R then $\mathcal{U}_i(A) = \mathcal{U}_i(A \mid \underline{b})$ for all i. Conversely, since R is Prüfer ring, the ideals $\mathcal{U}_i(A)$ are flat and consequently if $\mathcal{U}_i(A) = \mathcal{U}_i(A \mid \underline{b})$ from Corollary 3 it follows that (S) has a solution in R.

(i) $\Rightarrow$ (ii) We shall prove that if R is not a Prüfer ring then there exists a system (S): $A\underline{x} = \underline{b}$ which has no solution in R but for which $\mathcal{U}_i(A) = \mathcal{U}_i(A \mid \underline{b})$ for all i. If R is not a Prüfer ring note that by property (ii) of Prüfer rings one of the two following possibilities holds for R:

(1) There is a prime ideal $\mathfrak{p}_0$ of R such that $R_{\mathfrak{p}_0}$ is not a domain.

(2) For every prime ideal $\mathfrak{p}$ of R the ring $R_{\mathfrak{p}}$ is a domain but for some $\mathfrak{p}_0$, $R_{\mathfrak{p}_0}$ is not a Prüfer ring.

In the case (1), since $R_{\mathfrak{p}_0}$ is not a domain, there are two elements u, v in R such that $u/1$, $v/1$ are non zero in $R_{\mathfrak{p}_0}$ and $(u/1)\cdot(v/1)=0$ or equivalently $suv=0$ for some $s \notin \mathfrak{p}_0$. Now consider the system of linear equations:

$$(S)\colon \begin{cases} ux = 0, \\ svx = sv. \end{cases}$$

For (S) one has

$$\mathscr{U}_1(A) = \mathscr{U}_1(A \mid \underline{b}) = (u, sv),$$

$$\mathscr{U}_2(A) = \mathscr{U}_2(A \mid \underline{b}) = (0).$$

But the system $(S_{\mathfrak{p}_0})$ has no solution in $R_{\mathfrak{p}_0}$. In fact, for such a solution λ one would have $(u/1)\cdot\lambda = 0$. So $\lambda - 1$ is a unit in $R_{\mathfrak{p}_0}$. On the other hand, $(s/1)\cdot(v/1)\cdot\lambda = sv/1$ would imply $sv/1 = 0$ and hence $v/1 = 0$, which is contradictory. By Proposition 1 the system (S) has no solution in R and it is the required system in case (1).

In the case (2) we shall construct the system (S) by a similar procedure to that given in [5]. Since $R_{\mathfrak{p}_0}$ is not Prüfer, by applying Lemma 5, there exist finitely generated ideals $\mathfrak{a}', \mathfrak{b}', \mathfrak{c}'$ of $R_{\mathfrak{p}_0}$ such that

$$\mathfrak{a}' \cdot \mathfrak{b}' \subseteq \mathfrak{a}' \cdot \mathfrak{c}', \mathfrak{b}' \not\subset \mathfrak{c}'. \tag{2}$$

If $b' \in \mathfrak{b}'$, $b' \notin \mathfrak{c}'$ and $a' \in \mathfrak{a}'$, $a' \neq 0$, then one has

$$\mathfrak{a}' \cdot (a' \cdot b') \subseteq \mathfrak{a}' \cdot (a' \cdot \mathfrak{c}'), (a' \cdot b') \not\subset (a' \cdot \mathfrak{c}'),$$

and so we can assume without loss of generality that in Eq. (2) the ideal $\mathfrak{b}'$ is a principal ideal contained in $\mathfrak{a}'$.

Now take finitely generated ideals $\mathfrak{a}_1, \mathfrak{b}_1, \mathfrak{c}_1$ of R whose extensions to $R_{\mathfrak{p}_0}$ are, respectively, $\mathfrak{a}', \mathfrak{b}', \mathfrak{c}'$ and such that $\mathfrak{b}_1$ is a principal ideal. There exist $s_1, s_2 \notin \mathfrak{p}_0$ such that

$$(s_1 \cdot \mathfrak{a}_1) \cdot (s_2 \cdot \mathfrak{b}_1) \subseteq (s_1 \cdot \mathfrak{a}_1) \cdot \mathfrak{c}_1,$$

and that, moreover, the ideal $s_2 \cdot \mathfrak{b}_1$ is not contained in $\mathfrak{c}_1$ (otherwise one would have $\mathfrak{b}' \subset \mathfrak{c}'$ since $s_2 \notin \mathfrak{p}_0$). On the other hand, since $\mathfrak{b}' \subset \mathfrak{a}'$, there exist $s_3 \in R$, $s_3 \notin \mathfrak{p}_0$ such that

$$s_3 \cdot (s_2 \cdot \mathfrak{b}_1) \subseteq s_1 \cdot \mathfrak{a}_1.$$

Denoting by $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ the ideals $s_1 \cdot \mathfrak{a}_1$, $s_3 \cdot s_2 \cdot \mathfrak{b}_1$ and $s_3 \cdot \mathfrak{c}_1$, respectively, $\mathfrak{b}$ is a principal ideal one has

$$\mathfrak{a} \cdot \mathfrak{b} \subseteq \mathfrak{a} \cdot \mathfrak{c}, \mathfrak{b} \nsubseteq \mathfrak{c}, \mathfrak{b} \subseteq \mathfrak{a}.$$

If $a_1,..., a_n$ are generators for $\mathfrak{a}$, b is a generator for $\mathfrak{b}$ and $c_1,..., c_m$ are generators for $\mathfrak{c}$, then the linear system over R given by

$$(S): \begin{cases} a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = 0 \\ c_1 x_{n+1} + \cdots + c_m x_{n+m} = b, \end{cases}$$

has no solution in R since $b \notin \mathfrak{c}$. However one has

$$\mathscr{U}_1(A) = \mathscr{U}_1(A \mid \underline{b}) = \mathfrak{a} + \mathfrak{c},$$

$$\mathscr{U}_2(A) = \mathscr{U}_2(A \mid \underline{b}) = \mathfrak{a} \cdot \mathfrak{c}.$$

This completes the proof of case (2) and hence the theorem.

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