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Eratosthenes on the "Measurement" of the Earth

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In this paper it is argued that Eratosthenes's measurement of the earth depended on estimated distances and ratios as well as approximation procedures, and that precise observations were not involved. His method is reconstructed here from a number of ancient texts, and it is concluded that Cleomedes, or his source, misunderstood and misrepresented what Eratosthenes did. © 1984 Academic Press, Inc.

In diesem Aufsatz werden Argumente dafür vorgetragen, daß die Erdmessung des Erathosthenes auf geschätzten Entfernungen und Verhältnissen sowie auf Näherungsverfahren beruhte und daß präzise Beobachtungen nicht eingingen. Seine Methode wird anhand einer Anzahl von Quellen rekonstruiert. Als Schlußfolgerung ergibt sich, daß Cleomedes, oder seine Quelle, das Verfahren des Eratosthenes mißverstand bzw. mißverständlich darstellte. © 1984 Academic Press, Inc.

Cet article tend à montrer que la mesure de la Terre attribuée à Eratosthène ne dérive pas d'observations précises, mais se fonde sur une estimation des données et sur des procédés d'approximation. La reconstruction de la démarche à partir de divers textes antiques permet de conclure que Cléomède, ou sa source, a déformé la méthode d'Eratosthène. © 1984 Academic Press, Inc.

We are told by Cleomedes, in a story often retold, that Eratosthenes determined the size of the earth from the following data: (1) the distance between Syene and Alexandria is 5000 stades, and these two places lie on the same meridian; (2) at noon on summer solstice at Syene there is no shadow, i.e., the sun is directly overhead; and (3) at noon on summer solstice at Alexandria the shadow cast by a point of a gnomon in a bowl sundial (*skaphe*) reaches an arc equal to 1/50th of a circle from the base of the gnomon. By means of a simple geometric argument Eratosthenes calculated the circumference of the earth to be 250,000 stades (for the text of this passage, see, for example: [Thomas 1968, II, 266–273]; on Cleomedes, see also [Neugebauer 1941]).

This account is usually interpreted to mean that Eratosthenes was responsible for one of the earliest measurements on record, and a great deal of effort has been expended to discover the precise value of the stade that Eratosthenes used in order to compare his result with the modern value for the circumference of the earth (e.g., [Heath 1921, II, 107]). However, I take this approach to be misconceived and that no precise measurements were involved. The value 5000 stades is clearly a round number, perhaps based on a calculation of the number of days it took to march (or sail) from one place to the other, times an estimate of the average distance traversed in a day: rounded distances of this character are preserved in ancient geographical texts (cf. [Neugebauer 1975, 334, 1313]), and in all

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probability many of these values were already traditional by the time of Eratosthenes. This interpretation will be supported by an examination of the rest of the data in the report, and it shall be argued that Cleomedes, or his source, misrepresented what Eratosthenes did. The essential points to be shown are that the data in (3) were calculated and not observed, and that the reference to the *skaphe* is probably based on a misunderstanding of a geometric figure introduced to explain the procedure.

There are the following difficulties with the account in Cleomedes: (a) All earlier sources ascribe the value 252,000 stades (rather than 250,000 stades) to Eratosthenes for the circumference of the earth in passages that are textually secure ([Neugebauer 1975, 653]; cf. Strabo, *Geography,* ii.5.7, and ii.5.34; Vitruvius, *De architectura,* I.vi.9, ed. Granger, Vol. 1, pp. 60–61; Theon of Smyrna, *Expositio rerum mathematicarum,* ed. Hiller, 124.10–12; Heron, *Dioptra,* 36, ed. Schöne, 302.10–17; see also Geminus, *Elements of Astronomy,* ed. Manitius, 166.2: for full bibliographic citations, see [Neugebauer 1975, Vol. 3]). (b) It is said that Eratosthenes first arrived at the value 250,000 stades and later changed it to 252,000 stades so that 1° would equal 700 stades (cf. [Heath 1921, II,107]). This assumes that dividing a circle into 360° was the norm at the time of Eratosthenes, a point disputed by Neugebauer [1975, 671], who has argued that most of the early Greeks preferred to divide circles into sections of 30° (zodiacal signs), 15° (half-signs or "steps"), or $7\frac{1}{2}°$ ("parts"). (c) If $7\frac{1}{2}°$ is the unit commonly employed at the time, why would Eratosthenes introduce an arc of $7\frac{1}{5}°$ (360°/50)? Certainly no measurement of an arc on the surface of a bowl sundial with the tools available at the time could distinguish these two arcs. (d) If Cleomedes' source had the values 1/50th and 5000 stades, I take it that he arrived at the 250,000 stades simply by multiplying 50 times 5000, preferring that number to 252,000, which seemed to make no sense. But if the 1/50th and the 5000 are round numbers, it seems reasonable that the 252,000 stades might also be a value rounded to thousands of stades. (e) With the value 5000 stades for the distance from Alexandria to Syene, and a circumference of 252,000 stades, we find the following ratio:

$$\frac{5{,}000}{252{,}000} = \frac{1}{50\frac{2}{5}}. \qquad (1)$$

I suggest that Eratosthenes's 1/50th is also a round number; i.e., the ratio of the arc to the circumference of the circle is as 1 is to 50, rounded to an integer. One might be tempted to argue that the measured angle was $7\frac{1}{7}°$ (360°/50.4), but objection (c) seems to rule that out. For these reasons I claim that the 1/50th is a computed value and not a quantity measured directly.

I take it that Cleomedes or his source has misinterpreted a lost figure that originally accompanied Eratosthenes' demonstration, and I reconstruct it on the basis of a figure for constructing sundials in Vitruvius, *De architectura,* IX.vii (see Fig. 1). The "measured" quantity is the length of the shadow cast by a gnomon of known length at noon on summer solstice, and from the ratio of the two lengths it is required to find the corresponding arc. Note that it was quite common at the time to indicate geographical latitude by the ratio of small integers representing

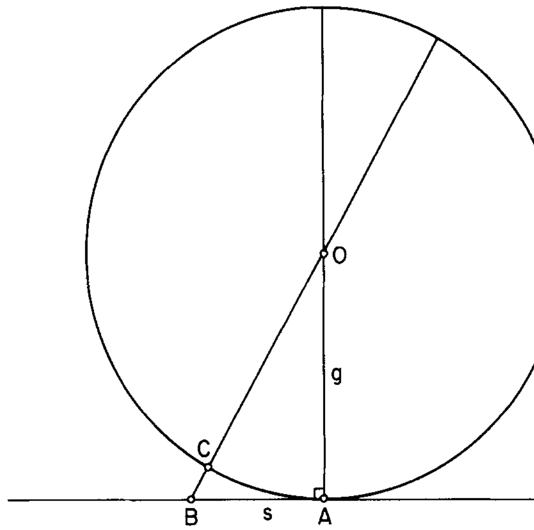

FIG. 1. A sundial in which OA is the gnomon of length g which casts a shadow on the horizontal line AB of length s when the sun is in the direction BO. The problem is to find the amount of arc AC from the ratio of s to g.

the lengths of the gnomon and the shadow at solstice and equinox (cf. [Neugebauer 1975, 746–748]); unfortunately only a corrupt value of this ratio is preserved for Alexandria in the early period (cf. [Neugebauer 1975, 336]). However, if we reconstruct the ratio of shadow to gnomon at noon, summer solstice, at Alexandria, accepted by Eratosthenes (possibly a traditional value) as 1 to 8, then the arc in question would be $7\frac{1}{8}°$, i.e.,

$$\arctan \tfrac{1}{8} = 7\tfrac{1}{8}°. \qquad (2)$$

Now $360°/7.125° = 50.53$, which would round to 51 by modern standards, but could be curtailed to 50 by ancient mathematicians. This leaves us with two problems: (i) How could Eratosthenes convert the ratio of shadow to gnomon into an arc; i.e., in modern terms, how could be evaluate the arctangent function? (ii) Is there a way to increase the ratio of shadow to gnomon such that the corresponding arc would be a bit more than 1/50.5 of a circle (i.e., closer to 1/50th of a circle)?

The strategy for (i) will be to find the tangent of 30°, and with the half-angle formula shown below the tangents of 15° and $7\frac{1}{2}°$ will be computed: this formula was probably known to Archimedes, a contemporary of Eratosthenes (cf. [Pedersen 1974, 60]). To do this we need a way to approximate square roots and a way to approximate the ratio of large numbers that are relatively prime by the ratio of small numbers. At that point we can invoke linear interpolation to find the arctangent sought.

In Fig. 2 let right triangle ABC be inscribed in a unit circle about center E, where h is the altitude on the hypotenuse, s is the side opposite A, and g is the side opposite C. If we let ED be x, then

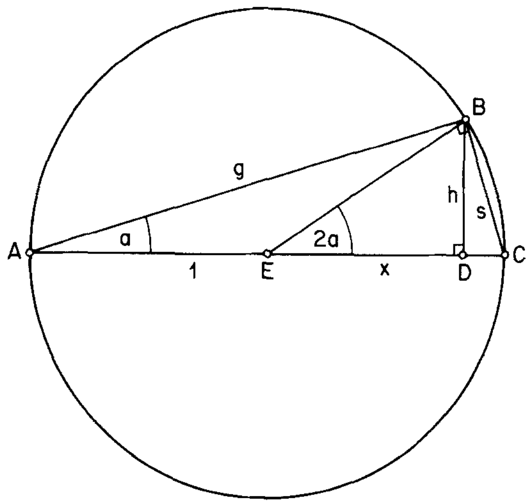

FIGURE 2

$$\frac{s}{g} = \frac{h}{1 + x} = \frac{1 - x}{h} \tag{3}$$

or

$$h^2 = 1 - x^2. \tag{4}$$

Let angle $2a$ be 30° (1/12th of a circle). Then

$$h/x = 1/\sqrt{3}. \tag{5}$$

If we approximate the square root by the formula

$$\sqrt{a^2 + b} \approx a + b/2a \tag{6}$$

and we substitute $\sqrt{27/9}$ for $\sqrt{3}$, then

$$\sqrt{3} = (\sqrt{25 + 2})/3 \approx 26/15.$$

Note that Archimedes used a much better approximation for the square root of 3 [Heath 1897, lxxx–xcix]. Hence the tangent of $2a$ (=30°) is

$$\tan 30° = h/x \approx 15/26. \tag{7}$$

In triangle ABC, where angle a is 15°,

$$\tan 15° = \frac{s}{g} = \frac{h}{1 + x} = \frac{1}{2 + \sqrt{3}} \approx \frac{15}{56}. \tag{8}$$

Applying the same argument again, letting angle $2a$ be 15°, we find from Eq. (8) that

$$h = x/(2 + \sqrt{3}). \tag{9}$$

Thus, from Eq. (4)

$$x = \sqrt{(7 + 4\sqrt{3})/(8 + 4\sqrt{3})}. \tag{10}$$

Substituting 26/15 for $\sqrt{3}$ yields

$$\begin{aligned} x &= \sqrt{209/224}, \\ h &= (15\sqrt{209/224})/56 \\ &= (15\sqrt{209 \times 224})/(56 \times 224). \end{aligned}$$

Hence, using Eq. (3),

$$\begin{aligned} \tan 7\tfrac{1}{2}^\circ = \frac{s}{g} = \frac{h}{1 + x} &\approx \frac{\sqrt{46816}}{56} - \frac{209}{56} \\ &\approx \frac{4\sqrt{2926} - 209}{56}. \end{aligned} \tag{11}$$

If we approximate $\sqrt{2926}$ by means of formula (6), above,

$$\begin{aligned} \sqrt{2926} &= \sqrt{54^2 + 10} \\ &\approx 54 + 10/(2 \times 54), \end{aligned}$$

then

$$\tan 7\tfrac{1}{2}^\circ = s/g \approx 199/1512. \tag{12}$$

To reduce the magnitude of the numerator one may use the method of continued fractions, presumed to be known at the time, but any method will do. That fractions of this kind were reduced in this sense is known from the works of Archimedes and Aristarchus (cf. [Heath 1897, 93–98; 1913, 336]; see also [Goldstein 1983; Fowler 1979]). Thus

$$s/g = 199/1512 \approx 5/38 \tag{13}$$

and this is a very good value for the tangent of $7\tfrac{1}{2}^\circ$, or 1/48th of a circle. If we start with the ratio

$$s/g = 1/8 \tag{14}$$

and seek y, the corresponding arc, then by linear interpolation

$$\frac{1/8}{5/38} = \frac{y}{1/48} \tag{15}$$

and

$$y = 19/960 \approx 1/50.5; \tag{16}$$

i.e., the arc corresponding to our assumed ratio for the shadow to gnomon is a little less than 1/50th of a circle. Note that 50.5 times 5000 stades is equal to 252,500 stades, which might have been rounded to 252,000 stades.

To increase the size of this arc as required to satisfy (ii), above, we could start with a ratio of the form

$$\frac{s}{g} = \frac{1}{8 - 1/n}. \tag{17}$$

If n lies between 28 and 200, an exact calculation yields values for the earth's circumference between 251,515 stades and 252,475 stades, all of which would properly be rounded to 252,000 stades, to the nearest thousand stades.

I conclude that most likely Eratosthenes began with a ratio for the shadow to gnomon for noon, summer solstice, at Alexandria of 1 to 8, but that possibly he made a small adjustment to that ratio. Then he computed the ratio of the corresponding arc to the circumference of the circle to be 1 to 50½, or thereabout. Using his approximate value of 5000 stades for the distance from Syene to Alexandria he then computed the circumference of the earth to be 252,000 stades rounded to thousands of stades. No precise measurements were required to reach this result.

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