Chapter 5

Differential equations

5.1 Ordinary and partial differential equations

A differential equation is a relation between an unknown function and its derivatives. Such equations are extremely important in all branches of science; mathematics, physics, chemistry, biochemistry, economics,...

Typical example are

- Newton's law of cooling which states that

  the rate of change of temperature is proportional to the temperature difference between it and that of its surroundings.

  This is formulated in mathematical terms as the differential equation

  $$\frac{dT}{dt} = k(T - T_0),$$

  where $T(t)$ is the temperature of the body at time t, T_0 the temperature of the surroundings (a constant) and k a constant of proportionality,

- the wave equation,

  $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},$$

  where $u(x, t)$ is the displacement (from a rest position) of the point x at time t and c is the wave speed.

The first example has unknown function T depending on one variable t and the relation involves the first order (ordinary) derivative $\frac{dT}{dt}$. This is a *ordinary differential equation*, abbreviated to ODE.

The second example has unknown function u depending on two variables x and t and the relation involves the second order partial derivatives $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial t^2}$. This is a *partial differential equation*, abbreviated to PDE.

The *order* of a differential equation is the order of the highest derivative that appears in the relation.

The unknown function is called the *dependent variable* and the variable or variables on which it depend are the *independent variables*.

A solution of a differential equation is an expression for the dependent variable in terms of the independent one(s) which satisfies the relation. The *general solution* includes all possible solutions and typically includes arbitrary constants (in the case of an ODE) or arbitrary functions (in the case of a PDE.) A solution without arbitrary constants/functions is called a *particular solution*. Often we find a particular solution to a differential equation by giving extra conditions in the form of initial or boundary conditions.

Example 5.1 Show that $\cos ct$ and $\sin ct$ are solutions of the second order ODE

$$\ddot{u} + c^2 u = 0,$$

where c is a constant.

Deduce that $A \cos ct + B \sin ct$ is also a solution for arbitrary constants A, B.

Remark   It is conventional to use $\dot{u}$ to denote the derivative (of u) with respect to t and $\ddot{u}$ the second derivative with respect to t.

In a similar way we will use u' and u'' to denotes derivatives with respect to x.

Solution   :

$\square$

Remarks

1. A differential equation which contains no products of terms involving the dependent variable is said to be *linear*. For example,

$$\frac{d^2 y}{dx} + x^2 = 0, \quad u_t + u_x = 0,$$

are linear but

$$\frac{d^2 y}{dx^2} + y^2 = 0, \quad u_t + u u_x = 0,$$

are *nonlinear*.

The ODE in the above example is also linear. As illustrated, any linear combination $(A \cos ct + B \sin ct)$ of given solutions $(\cos ct, \sin ct)$ is also a solution. This is true for all *linear* differential equations and makes them much easier to solve. It is *not* true of nonlinear differential equations.

2. The solution $u = A \cos ct + B \sin ct$ contains two arbitrary constants and is the general solution of the second order ODE $\ddot{u} + c^2 u = 0$. This illustrates the fact that the general solution of an nth order ODE contains n arbitrary constants.

5.2 First order ODEs

We will study methods for solving first order ODEs which have one of three special forms.

Separable type[1]

Consider first, for example, the ODE

$$\frac{dy}{dx} = x.$$

This is precisely the same as writing

$$y = \int x \, dx = \tfrac{1}{2}x^2 + C,$$

where C is an arbitrary constant—the constant of integration. This is the general solution of the ODE.

More generally, ODEs of the form

$$\frac{dy}{dx} = f(x)g(y),$$

are called *separable* and can be solved in a similar way. Dividing by $g(y)$ and integrating both sides with respect to x we get

$$\int \frac{1}{g(y)} \frac{dy}{dx} \, dx = \int f(x) \, dx.$$

Recalling the formula for integration by change of variables, we see that the integral on the left is equal to

$$\int \frac{1}{g(y)} \, dy.$$

Hence the separable ODE is equivalent to the relationship between integrals

$$\int \frac{1}{g(y)} \, dy = \int f(x) \, dx.$$

Assuming that these integrals may be evaluated, we get

$$G(y) = F(x) + C,$$

where C is an arbitrary constant. This gives the general solution.

Example 5.2 Find the general solution of

$$y' = e^{x+4y},$$

and the particular solution for which $y(0) = 0$.

Solution :

---

[1] This type of ODE is studied in level-1 modules.

Answer: The general solution,

$$y = -\tfrac{1}{4}\log(-4(e^x + C)) = -\tfrac{1}{4}\log(C - 4e^x),$$

where we have relabelled the arbitrary constant C $(= -4 \times$ old value of $C)$.
The particular solution is

$$y = -\tfrac{1}{4}\log(5 - 4e^x).$$

□

Remark The same techniques may also be used to solve PDEs which are separable in the same sense. For example, the PDE

$$y\frac{\partial u}{\partial x} = u^2,$$

only has a derivative with respect to x and so we regard y as fixed and rewrite it as

$$\int_{y \text{ fixed}} y\frac{du}{u^2} = \int_{y \text{ fixed}} dx.$$

Hence

$$-y\frac{1}{u} = x + A(y), \quad \text{i.e., } u = \frac{-y}{x + A(y)}.$$

where the "constant of integration" is the arbitrary function $A(y)$. This must, in general, depend on y since this variable was fixed during the integration.

Exact type

An ODE of the form

$$\frac{d}{dx}\phi(x, y) = 0,$$

is said to be *exact* and obviously has the general solution $\phi(x, y) = C$, where C is an arbitrary constant. Being able to recognise that an ODE can be expressed in this form is more difficult however.

Using the chain rule for functions of two variables, we have

$$\frac{d}{dx}\phi(x, y) = \frac{\partial \phi}{\partial x} + \frac{dy}{dx}\frac{\partial \phi}{\partial y},$$

(see Example **??**). Hence an ODE of the form

$$P(x, y) + Q(x, y)\frac{dy}{dx} = 0,$$

is exact provided there exists some $\phi(x, y)$ such that

$$P = \frac{\partial \phi}{\partial x} \text{ and } Q = \frac{\partial \phi}{\partial y}.$$

For example, the ODE

$$y + x\frac{dy}{dx} = 0,$$

is exact since we can take $\phi = xy$ giving

$$\frac{\partial \phi}{\partial x} = y \text{ and } \frac{\partial \phi}{\partial y} = x.$$

In general, exact ODEs are characterised by the following theorem.

Theorem *The ODE*

$$P(x, y) + Q(x, y)\frac{dy}{dx} = 0,$$

is exact if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

Proof If the ODE is exact then there exists ϕ such that

$$P = \frac{\partial \phi}{\partial x} \text{ and } Q = \frac{\partial \phi}{\partial y},$$

and hence it is *necessary* that

$$\frac{\partial P}{\partial y} = \frac{\partial^2 \phi}{\partial y \partial x} = \frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial Q}{\partial x}.$$

The proof that this condition is enough to guarantee the existence of ϕ is omitted. $\qquad\square$

Example 5.3 Show that the ODE

$$y + \cos(x + y) + \big(x - y + \cos(x + y)\big)\frac{dy}{dx} = 0,$$

is exact an find its general solution.

Solution :

Answer: The general solution is

$$xy + \sin(x + y) - \tfrac{1}{2}y^2 = C,$$

where C is an arbitrary constant. □

Integrating factors

The differential equation

$$P + Q\frac{dy}{dx} = 0, \tag{1}$$

has the same solutions as the one obtained by multiplying through by a factor $\mu(x, y)$

$$(\mu P) + (\mu Q)\frac{dy}{dx} = 0. \tag{2}$$

This opens up the possibility that, by multiplying by an *integrating factor* μ, we may convert a non-exact ODE (1), for which

$$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x},$$

into an exact one (2), for which

$$\frac{\partial}{\partial y}(\mu P) = \frac{\partial}{\partial x}(\mu Q).$$

An important special case of this is covered at level-1. Recall that the *linear* first order ODE

$$\frac{dy}{dx} + a(x)y = b(x),$$

has an integrating factor

$$\mu = \exp\left(\int a\,dx\right).$$

The ODE may be written as

$$\frac{d}{dx}(\mu y) = \mu b,$$

and the general solution obtained by integration.

Example 5.4 Find the general solution of

$$x\frac{dy}{dx} - 2y = 2x^5.$$

Solution :

Answer: The general solution is

$$y = x^2 \int 2x^2 \, dx = x^2 (\tfrac{2}{3}x^3 + C) = \tfrac{2}{3}x^5 + Cx^2.$$

□

Example 5.5 Show that the ODE

$$2y + 2x^2y^2 + (x + x^3y)\frac{dy}{dx} = 0,$$

is not exact.

Find an integrating factor x^n (for some n, to be determined) and hence find its general solution.

Solution :

55

Answer: The general solution is

$$\phi = x^2 y + \tfrac{1}{2} x^4 y^2 = C.$$

$\square$

Example 5.6 Solve the ODE

$$\frac{dy}{dx} = \frac{4xy + 5y^3}{4x^2 - 2y^3},$$

by finding an integrating factor depending only on y.

Solution :

Answer: The general solution is

$$\phi = \frac{2x^2}{y^2} + 5x + 2y = C.$$

$\square$

5.3   Second order ODEs

We will only consider *linear* second order ODEs, those of the form

$$a(x)\frac{d^2 y}{dx^2} + b(x)\frac{dy}{dx} + c(x)y = f(x). \tag{1}$$

If the right hand side is identically zero, we say that the ODE is *homogeneous*. More generally, there is a homogeneous ODE associated with each ODE of the form (1), obtained by replacing $f(x)$ by 0,

$$a(x)\frac{d^2y}{dx^2} + b(x)\frac{dy}{dx} + c(x)y = 0. \tag{2}$$

If the *general* solution of the homogeneous form (2) is known, the general solution of the full ODE (1) may be found using the following result.

Theorem *The general solution of the ODE*

$$a(x)\frac{d^2y}{dx^2} + b(x)\frac{dy}{dx} + c(x)y = f(x),$$

is y = CF + PI, where CF is the general solution of homogenous form

$$a(x)\frac{d^2y}{dx^2} + b(x)\frac{dy}{dx} + c(x)y = 0,$$

called the complementary function *and PI is* any *solution of the full ODE, called a* particular integral.

If $f(x)$ is simple, it is often possible to guess a particular integral. This is the method used in level-1 courses and an application of this result is given in Example **??** (b). See also Adams §17.8, p1016.

However, if $f(x)$ is itself a solution of the homogeneous ODE then it can be difficult to guess a particular integral. In these circumstances a more powerful technique is needed. This is the method of *reduction of order*.

Reduction of order

Given one solution u of (2), the general solution of (1) may be found by solving a linear first order ODE. This process is known as *reduction of order*.

Let u be a solution of (2), i.e., u is such that

$$au'' + bu' + cu = 0. \tag{3}$$

Now set $y = uv$ in (1), where v will be the new dependent variable in the ODE, to obtain

$$a(uv)'' + b(uv)' + c(uv) = f.$$

Using the product rule and regrouping terms, this becomes

$$(au'' + bu' + cu)v + 2au'v' + auv'' + buv' = f.$$

Since u satisfies (3), we get

$$auv'' + (bu + 2au')v' = f.$$

Finally, we replace v' by w to get the first order linear ODE

$$auw' + (bu + 2au')w = f,$$

where a, b, f are known from the original ODE (1) and u is the known solution of (2). This is a linear first order ODE, which may be solved by the method demonstrated in Example **??** .

In summary, we may solve (1) by the following method (if we know a solution u of (1))

1. replace y in (1) by uv and determine the ODE satisfied by v; there is no term in v itself

2. replace v' in this ODE by w to obtain a linear first order ODE

3. find the general solution w of this ODE (by finding an integrating factor etc.)

4. integrate $w = v'$ to obtain v,

5. the general solution of (1) is $y = uv$.

Euler equation

Euler equation are of the form

$$ax^2y'' + bxy' + cy = s(x), \tag{4}$$

where a, b, c are constants and $a \neq 0$. One way of solving Euler equations is to substitute $y = x^k$, where k is a real number, into the corresponding homogeneous equation,

$$ax^2y'' + bxy' + cy = 0,$$

and solve to find k. You can now apply the Reduction of order method to the original equation (4) by setting $y = uv = x^k v$.

Example 5.7 Find the general solution of

$$x^2y'' + 3xy' + y = x. \tag{1}$$

Solution :

Answer: $y = \frac{1}{x}v = \frac{1}{4}x + C\frac{\log|x|}{x} + \frac{D}{x}$, which is the general solution of (1). □

Constant coefficients

The ODE
$$ay'' + by' + cy = 0,$$

where a, b, c are constants (with $a \neq 0$) always has a solution of the form $y = e^{\lambda t}$, where λ is a constant (which may be real or complex), to be determined.

Making this substitution in the ODE, it is found that λ must satisfy the *auxiliary equation*

$$a\lambda^2 + b\lambda + c = 0,$$

a quadratic equation which may be solved to find λ. There are three case

Distinct real solutions λ, μ The general solution of the homogeneous equation is $y = Ae^{\lambda x} + Be^{\mu x}$.

One real (repeated) solutions λ One solution $e^{\lambda t}$ is known. Reduction of order is used to find the general solution $y = (A + Bx)e^{\lambda x}$.

Complex roots $\alpha \pm i\beta$ The general solution of the homogeneous equation is

$$y = A'e^{(\alpha + i\beta)t} + B'e^{(\alpha - i\beta)t} = e^{\alpha t}(A'e^{i\beta t} + B'e^{-i\beta t}) = e^{\alpha t}(A\cos \beta t + B\sin \beta t),$$

where $A = A' + B'$ and $B = iA' - iB'$.

In all case, for an inhomogeneous equation, reduction of order may be used to find the general solution.

Example 5.8 Find the general solutions of

(a) $y'' + 2y' - 3y = 0$, (b) $y'' - 3y' + 2y = e^{-x}$, (c) $y'' + 2y' + 3y = 0$, (d) $y'' + 2y' + y = x$.

Solution :

Answer:

(a) The general solution is $y = Ae^{-3x} + Be^x$, where A, B are arbitrary.

(b) The general solution is $y = Ae^x + Be^{2x} + \frac{1}{6}e^{-x}$.

(c) $\lambda = -1 \pm i\sqrt{2}$, and so the general solution is $y = e^{-x}(A\cos\sqrt{2}x + B\sin\sqrt{2}x)$.

(d) The general solution of the original ODE is $y = e^{-x}v = x - 2 + Cxe^{-x} + De^{-x}$. $\qquad\square$

5.4 Partial differential equations

In this section certain first and second order PDEs will be solved by means of a change of variables. Although there is a theory which may be used to *determine* the appropriate change of variable (see Mathematics 3H PDEs or 3S), in this module the change of variable will always be given.

We will make a change of independent variables from x, y to u, v (say). If $z = z(x, y)$ and we introduce new variables $u = u(x, y)$, $v = v(x, y)$ then the chain rule gives

$$\frac{\partial z}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial z}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial z}{\partial v}.$$

More generally, this shows that for *any* expression $*$ that is to be thought of as a function of x, y or of u, v,

$$\frac{\partial}{\partial x}(*) = \frac{\partial u}{\partial x}\frac{\partial}{\partial u}(*) + \frac{\partial v}{\partial x}\frac{\partial}{\partial v}(*). \tag{1}$$

This general form of the chain rule is useful when calculating second order derivatives.

First order PDEs

Example 5.9 By changing variables from (x, y) to (u, v), where $u = xy$, $v = x/y$, solve the PDE

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2x^2\sin(xy).$$

Solution Answer: $z = -\frac{x}{y}\cos(xy) + A\left(\frac{x}{y}\right)$, where A is an arbitrary function. $\square$

Example 5.10 Solve the PDEs

$$\text{(a) } f_x + f_y = f, \quad \text{(b) } f_x + f_y = f + y,$$

by considering the change of variables $(x, y) \to (u, v)$ where $u = x - y$, $v = x$.

Solution :

Answer: (a) $\log|f| = v + A(u)$, and so $f = \pm e^{A(u)}e^v = B(u)e^v$. In terms of x and y this is

$$f = B(x - y)e^x,$$

where B is an arbitrary function.

(b)$f = e^v\left(-ve^{-v} - e^{-v} + ue^{-v} + A(u)\right) = u - v - 1 + A(u)e^v = -y - 1 + A(x - y)e^x$, where A is an arbitrary function. $\qquad\square$

Second order PDEs

The application of the chain rule to calculating second derivatives is a bit more complicated. We consider the second x-derivative as an example.

Now, we use the product rule to obtain

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2}\frac{\partial z}{\partial u} + \frac{\partial u}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + \frac{\partial^2 v}{\partial x^2}\frac{\partial z}{\partial v} + \frac{\partial v}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right). \qquad (2)$$

In this there are two expressions which may be calculated using the general form of the chain rule (1); first, replacing $*$ by $\dfrac{\partial z}{\partial u}$,

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) = \frac{\partial u}{\partial x}\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right) + \frac{\partial v}{\partial x}\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right) = \frac{\partial u}{\partial x}\frac{\partial^2 z}{\partial u^2} + \frac{\partial v}{\partial x}\frac{\partial^2 z}{\partial v\partial u},$$

and replacing $*$ by $\dfrac{\partial z}{\partial v}$,

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right) = \frac{\partial u}{\partial x}\frac{\partial^2 z}{\partial u\partial v} + \frac{\partial v}{\partial x}\frac{\partial^2 z}{\partial v^2}.$$

Finally, these expression can be substituted into (2) to get the final answer. The following example illustrates this technique.

Example 5.11 Let $u = x$ and $v = y/x$. Determine w_{xy} in terms of u, v and derivatives of w with respect to u and v.

Solution :

Amswer: $w_{xy} = -\frac{1}{u^2} w_v + \frac{1}{u} \left(w_{uv} - \frac{v}{u} w_{vv} \right)$. □

Now we consider two examples where this technique is used to reduce a PDE to a simple form and then solve it.

Example 5.12 Solve the PDE
$$x^2 z_{xx} + 2xy z_{xy} + y^2 z_{yy} = 0,$$
by using the change of variables
$$u = x, \quad v = \frac{y}{x}.$$

Solution :

Answer: $z = xA\left(\frac{y}{x}\right) + B\left(\frac{y}{x}\right).$ □

Example 5.13 Solve the PDE

$$x^2 z_{xx} - y^2 z_{yy} = yz_y - xz_x,$$

by using the change of variables

$$u = xy, \quad v = \frac{y}{x}.$$

Solution :

Answer: $z(x, y) = B(xy) + C\left(\frac{y}{x}\right).$ □

<p style="text-align:center">End of 2X-Course</p>