Complex Numbers

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Abstract

This article discusses some introductory ideas associated with complex numbers, their algebra and geometry. This includes a look at their importance in solving polynomial equations, how complex numbers add and multiply, and how they can be represented. Finally we look at the nth roots of unity, that is, the solutions of the equations $z^n = 1$.

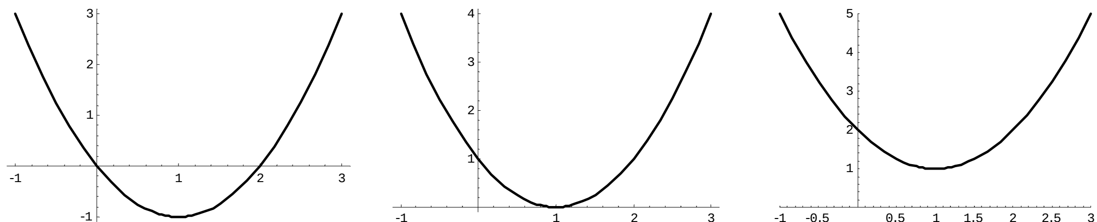
1 The Need For Complex Numbers

The shortest path between two truths in the real domain passes through the complex domain
Jacques Hadamard (1865-1963)

All of you will know that the two roots of the equation $ax^2 + bx + c = 0$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{1}$$

and solving quadratic equations is something that mathematicians have been able to do since the time of the Babylonians. When $b^2 - 4ac > 0$ then these two roots are real and distinct; graphically they are where the curve $y = ax^2 + bx + c$ cuts the x-axis. When $b^2 - 4ac = 0$ then we have one real root and the curve just touches the x-axis here. But what happens when $b^2 - 4ac < 0$? In this case there are no real solutions to the equation, as no real number squares to give the negative $b^2 - 4ac$. From the graphical point of view, the curve $y = ax^2 + bx + c$ lies entirely above or below the x-axis.



Two Distinct Real Roots — One Repeated Real Root — No Real Roots

It is only comparatively recently that mathematicians have been comfortable with these roots when $b^2 - 4ac < 0$. During the Renaissance the quadratic would have been considered unsolvable, or its roots would have been called *imaginary*[1].

If we imagine $\sqrt{-1}$ to exist, and that it behaves much like other numbers, then the two roots of the quadratic $ax^2 + bx + c = 0$ can be written in the form

$$x = A \pm B\sqrt{-1} \tag{2}$$

*These pages are produced by Richard Earl, who is the Schools Liaison and Access Officer for mathematics, statistics and computer science at Oxford University. Any comments, suggestions or requests for other material would be welcomed at earl@maths.ox.ac.uk

[1] The term 'imaginary' was first used by the French Mathematician René Descartes (1596-1650). Whilst he is known more as a philosopher, Descartes made many important contributions to mathematics and helped found co-ordinate geometry — hence the naming of Cartesian co-ordinates.

where $A = -b/2a$ and $B = \sqrt{4ac - b^2}/2a$ are real numbers. But what meaning can such roots have? It was this philosophical point which pre-occupied mathematicians until the start of the 19th century; afterwards these 'imaginary' numbers started proving so useful (especially in the work of Cauchy and Gauss) that these philosophical concerns were essentially forgotten.

Notation 1 *We shall from now on write i for $\sqrt{-1}$ — this is standard notation amongst mathematicians*[2]*, though many books, particularly those written for engineers and physicists, use j instead.*

Definition 2 *A complex number*[3] *is a number of the form $a + bi$ where a and b are real numbers. If $z = a + bi$ then a is known as the real part of z and b as the imaginary part. We write $a = \operatorname{Re} z$ and $b = \operatorname{Im} z$. Note that real numbers are complex — a real number is simply a complex number with zero imaginary part.*

Remark 3 *Note that two complex numbers are equal precisely when their real and imaginary parts are equal — that is $a+bi = c+di$ if and only if $a = c$ and $b = d$. This is called 'comparing real and imaginary parts'.*

Notation 4 *We write $\mathbb{C}$ for the set of all complex numbers.*

One of the first major results concerning complex numbers, and which conclusively demonstrated their usefulness, was proved by Gauss in 1799. From the quadratic formula (1) we know that all quadratic equations can be solved using complex numbers, but what Gauss was the first to prove was the much more general result:

Theorem 5 *(FUNDAMENTAL THEOREM OF ALGEBRA) The roots of any polynomial equation*

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0,$$

with real (or complex) coefficients a_i, are complex. That is there are n (not necessarily distinct) complex numbers $\gamma_1, \ldots, \gamma_n$ such that

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = a_n (x - \gamma_1)(x - \gamma_2) \cdots (x - \gamma_n).$$

In particular, this shows that a degree n polynomial has, counting repetitions, exactly n roots in $\mathbb{C}$.

The proof of this theorem is far beyond the scope of this document. Note that the theorem only guarantees the *existence* of the roots of a polynomial somewhere in $\mathbb{C}$ unlike the quadratic formula, which plainly gives us a formula for the roots. The theorem gives no hint as to where in $\mathbb{C}$ these roots are to be found.

Exercise 1 ***A.*** *Which of the following quadratic equations require the use of complex numbers to solve them?*

$$3x^2 + 2x - 1 = 0, \quad 2x^2 - 6x + 9 = 0, \quad -4x^2 + 7x - 9 = 0.$$

Exercise 2 ***B.*** *On separate axes, sketch the graphs of the following cubics, being sure to carefully label any turning points. In each case state how many of the cubic's roots are real.*

$$\begin{aligned} y_1(x) &= x^3 - x^2 - x + 1; \\ y_2(x) &= 3x^3 + 5x^2 + x + 1; \\ y_3(x) &= -2x^3 + x^2 - x + 1. \end{aligned}$$

[2] The i notation was first introduced by the Swiss mathematician Leonhard Euler (1707-1783). Much of our modern notation is due to him including e and π. Euler was a giant in 18th century mathematics and the most prolific mathematician ever. His most important contributions were in analysis (eg. on infinite series, calculus of variations). The study of topology arguably dates back to his solution of the *Königsberg Bridge Problem.*

[3] The term 'complex number' is due to the German mathematician Carl Gauss (1777-1855). Gauss is considered by many the greatest mathematician ever. He made major contributions to almost every area of mathematics from number theory and non-Euclidean geometry, to astronomy and magnetism. His name precedes a multitude of theorems and definitions throughout mathematics.

Exercise 3 ***C.*** *Let p and q be real numbers with $p \leq 0$. Find the co-ordinates of the turning points of the cubic $y = x^3 + px + q$. Show that the cubic equation $x^3 + px + q = 0$ has three real roots, with two or more repeated, precisely when*

$$4p^3 + 27q^2 = 0.$$

Under what conditions on p and q does $x^3 + px + q = 0$ have (i) three distinct real roots, (ii) just one real root? How many real roots does the equation $x^3 + px + q = 0$ have when $p > 0$?

Exercise 4 ***C.*** *By making a substitution of the form $X = x - \alpha$ for a certain choice of α, transform the equation $X^3 + aX^2 + bX + c = 0$ into one of the form $x^3 + px + q = 0$. Hence find conditions under which the equation*

$$X^3 + aX^2 + bX + c = 0$$

has (i) three distinct real roots, (ii) three real roots involving repetitions, (iii) just one real root.

Exercise 5 ***C*** *The cubic equation $x^3 + ax^2 + bx + c = 0$ has roots α, β, γ so that*

$$x^3 + ax^2 + bx + c = (x - \alpha)(x - \beta)(x - \gamma).$$

By equating the coefficients of powers of x in the previous equation, find expressions for a, b and c in terms of α, β and γ.

Given that α, β, γ are real, what can you deduce about their signs if (i) $c < 0$, (ii) $b < 0$ and $c < 0$, (iii) $b < 0$ and $c = 0$.

Exercise 6 ***C.*** *With a, b, c and α, β, γ as in the previous exercise, let $S_n = \alpha^n + \beta^n + \gamma^n$. Find expressions S_0, S_1 and S_2 in terms of a, b and c. Show further that*

$$S_{n+3} + aS_{n+2} + bS_{n+1} + cS_n = 0$$

for $n \geq 0$ and hence find expressions for S_3 and S_4 in terms of a, b and c.

2 Basic Operations

We add, subtract, multiply and divide complex numbers much as we would expect. We add and subtract complex numbers by adding their real and imaginary parts:-

$$\begin{aligned} (a + bi) + (c + di) &= (a + c) + (b + d)\,i, \\ (a + bi) - (c + di) &= (a - c) + (b - d)\,i. \end{aligned}$$

We can multiply complex numbers by expanding the brackets in the usual fashion and using $i^2 = -1$,

$$(a + bi)(c + di) = ac + bci + adi + bdi^2 = (ac - bd) + (ad + bc)\,i.$$

To divide complex numbers, we note firstly that $(c + di)(c - di) = c^2 + d^2$ is real. So

$$\frac{a + bi}{c + di} = \frac{a + bi}{c + di} \times \frac{c - di}{c - di} = \left(\frac{ac + bd}{c^2 + d^2}\right) + \left(\frac{bc - ad}{c^2 + d^2}\right) i.$$

The number $c - di$, which we just used, as relating to $c + di$, has a special name and some useful properties — see Proposition 10.

Definition 6 *Let $z = a + bi$. The conjugate of z is the number $a - bi$, and this is denoted as $\bar{z}$ (or in some books as z^*).*

Note from equation (2) that when the *real* quadratic equation $ax^2 + bx + c = 0$ has complex roots, then these roots are conjugates of each other. Generally, if the polynomial $a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 = 0$, where the a_i are real, has a root z_0, then the conjugate $\bar{z}_0$ is also a root.

Exercise 7 ***A.*** *Put each of the following numbers into the form $a + bi$.*

$$(1+2i)(3-i), \quad \frac{1+2i}{3-i}, \quad (1+i)^4.$$

Exercise 8 ***A.*** *Let $z_1 = 1+i$ and let $z_2 = 2-3i$. Put each of the following into the form $a+bi$.*

$$z_1 + z_2, \quad z_1 - z_2, \quad z_1 z_2, \quad z_1/z_2, \quad \bar{z}_1 \bar{z}_2.$$

We needed a special symbol i for $\sqrt{-1}$, but we see now that further symbols are needed to find the square root of i. In fact we already knew this from the Fundamental Theorem, which implies that $z^2 = i$ has two roots amongst the complex numbers. The quadratic formula (1), is also valid for complex coefficients a, b, c, provided that proper sense is made of the square roots of the complex number $b^2 - 4ac$.

Problem 7 *Find all those z that satisfy $z^2 = i$.*

Suppose that $z^2 = i$ and $z = a + bi$, where a and b are real. Then

$$i = (a+bi)^2 = \left(a^2 - b^2\right) + 2abi.$$

Comparing the real and imaginary parts (see Remark 3), we know that

$$a^2 - b^2 = 0 \text{ and } 2ab = 1.$$

So $b = 1/2a$ from the second equation, and substituting for b into the first equation gives $a^4 = 1/4$, which has *real* solutions $a = 1/\sqrt{2}$ or $a = -1/\sqrt{2}$.

So the two z which satisfy $z^2 = i$, i.e. the two square roots of i, are

$$\frac{1+i}{\sqrt{2}} \text{ and } \frac{-1-i}{\sqrt{2}}.$$

Problem 8 *Use the quadratic formula to find the two solutions of*

$$z^2 - (3+i)\,z + (2+i) = 0.$$

We see that $a = 1$, $b = -3-i$, and $c = 2+i$. So

$$b^2 - 4ac = (-3-i)^2 - 4 \times 1 \times (2+i) = 9 - 1 + 6i - 8 - 4i = 2i.$$

Knowing $\sqrt{i} = \pm\,(1+i)\,/\sqrt{2}$, from the previous problem, we have

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2-4ac}}{2a} \\ &= \frac{(3+i) \pm \sqrt{2i}}{2} \\ &= \frac{(3+i) \pm \sqrt{2}\sqrt{i}}{2} \\ &= \frac{(3+i) \pm (1+i)}{2} \\ &= \frac{4+2i}{2} \text{ or } \frac{2}{2} \\ &= 2+i \text{ or } 1. \end{aligned}$$

Exercise 9 ***A.*** *Find the square roots of $-5-12i$, and hence solve the quadratic equation*

$$z^2 - (4+i)\,z + (5+5i) = 0.$$

Exercise 10 ***B.*** *Show that the complex number* $1+i$ *is a root of the cubic equation*

$$z^3 + z^2 + (5-7i)\,z - (10+2i) = 0,$$

and hence find the other two roots.

Exercise 11 ***B.*** *Show that the complex number* $2+3i$ *is a root of the quartic equation*

$$z^4 - 4z^3 + 17z^2 - 16z + 52 = 0,$$

and hence find the other three roots.

Exercise 12 ***B.*** *Let* n *be a positive integer. Simplify the expression* $(1+i)^{2n}$. *Use the binomial theorem to show that*

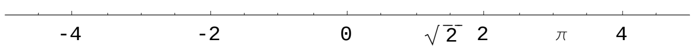
$$\binom{2n}{0} - \binom{2n}{2} + \binom{2n}{4} - \binom{2n}{6} + \cdots + (-1)^n \binom{2n}{2n} = \begin{cases} (-1)^{n/2}\, 2^n & \text{if } n \text{ is even;} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Show that the right-hand side is equal to $2^n \cos(n\pi/2)$. *Similarly, find the value of*

$$\binom{2n}{1} - \binom{2n}{3} + \binom{2n}{5} - \binom{2n}{7} + \cdots + (-1)^{n-1} \binom{2n}{2n-1}.$$

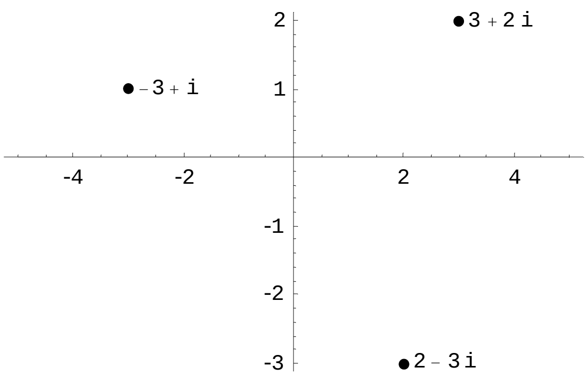
3 The Argand Diagram

The real numbers are often represented on the *real line* which increase as we move from left to right.



The Real Line

The complex numbers, having two components, their real and imaginary parts, can be represented as a plane; indeed, $\mathbb{C}$ is sometimes referred to as the *complex plane,* but more commonly, when we represent $\mathbb{C}$ in this manner, we call it an *Argand diagram*[4].The point (a,b) represents the complex number $a+bi$ so that the x-axis contains all the real numbers, and so is termed the *real axis,* and the y-axis contains all those complex numbers which are purely imaginary (i.e. have no real part), and so is referred to as the *imaginary axis.*



The Argand Diagram

[4] After the Swiss mathematician Jean-Robert Argand (1768-1822).

We can think of $z_0 = a + bi$ as a point in an Argand diagram, but it can often be useful to think of it as a vector as well. Adding z_0 to another complex number translates that number by the vector $\binom{a}{b}$. That is the map $z \mapsto z + z_0$ represents a translation a units to the right and b units up in the complex plane. Note that the conjugate $\bar{z}$ of a point z is its mirror image in the real axis. So, $z \mapsto \bar{z}$ represents reflection in the real axis.

Exercise 13 ***B.*** *Multiplication by i takes the point $x+iy$ to the point $-y+ix$. What transformation of the Argand diagram does this represent? What is the effect of multiplying a complex number by $(1+i)/\sqrt{2}$? [Hint: recall that this is square root of i.]*

A complex number z in complex plane can be represented by Cartesian co-ordinates, its real and imaginary parts, but equally useful is the representation of z by polar co-ordinates. If we let r be the distance of z from the origin, and if, for $z \neq 0$, we define θ to be the angle that the line connecting the origin to z makes with the positive real axis, then we can write

$$z = x + iy = r\cos\theta + ir\sin\theta. \tag{3}$$

The relations between z's Cartesian and polar co-ordinates are simple — we see that

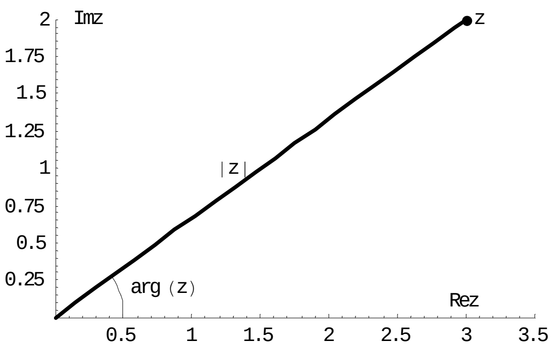
$$\begin{aligned} x &= r\cos\theta \quad \text{and} \quad y = r\sin\theta, \\ r &= \sqrt{x^2+y^2} \quad \text{and} \quad \tan\theta = \frac{y}{x}. \end{aligned}$$

Definition 9 *The number r is called the modulus of z and is written $|z|$. The number θ is called the argument of z and is written $\arg z$. If $z = x + iy$ then*

$$|z| = \sqrt{x^2+y^2} \quad \textit{and} \quad \sin\arg z = \frac{y}{\sqrt{x^2+y^2}}, \quad \cos\arg z = \frac{x}{\sqrt{x^2+y^2}}.$$

Note that the argument of 0 *is undefined.*

Note that $\arg z$ is defined only up to multiples of 2π. For example, the argument of $1+i$ could be $\pi/4$ or $9\pi/4$ or $-7\pi/4$ etc.. For simplicity, in this article we shall give all arguments in the range $0 \leq \theta < 2\pi$, so that $\pi/4$ would be the preferred choice here.



A Complex Number's Cartesian and Polar Co-ordinates

Exercise 14 ***A.*** *Find the modulus and argument of each of the following numbers.*

$$1+\sqrt{3}i, \quad (2+i)(3-i), \quad (1+i)^5.$$

Exercise 15 ***B.*** *Let α be a real number in the range $0 < \alpha < \pi/2$. Find the modulus and argument of the following numbers.*

$$\cos\alpha - i\sin\alpha, \quad \sin\alpha - i\cos\alpha, \quad 1 + i\tan\alpha, \quad 1 + \cos\alpha + i\sin\alpha.$$

Exercise 16 ***B.*** *On separate Argand diagrams sketch the following sets:*

(i)	$\lvert z\rvert < 1$;	*(ii)*	$\operatorname{Re} z = 3$;	*(iii)*	$\lvert z-1\rvert = \lvert z+i\rvert$;
(iv)	$-\pi/4 < \arg z < \pi/4$;	*(v)*	$\operatorname{Re}(z+1) = \lvert z-1\rvert$;	*(vi)*	$\arg(z-i) = \pi/2$;
(vii)	$\lvert z-3-4i\rvert = 5$;	*(viii)*	$\operatorname{Re}((1+i)z) = 1$.	*(ix)*	$\operatorname{Im}\left(z^3\right) > 0$.

We now prove some useful algebraic properties of the modulus, argument and conjugate functions.

Proposition 10 *Let $z, w \in \mathbb{C}$. Then*

$$\begin{array}{ll}
|zw| = |z|\,|w|\,; & |z/w| = |z|\,/\,|w| \quad \text{if} \quad w \neq 0; \\
z\bar{z} = |z|^2\,; & |\bar{z}| = |z|\,; \\
\overline{z+w} = \bar{z} + \bar{w}; & \overline{z-w} = \bar{z} - \bar{w}; \\
\overline{zw} = \bar{z}\ \bar{w}; & \overline{z/w} = \bar{z}/\bar{w} \quad \text{if} \quad w \neq 0; \\
|z+w| \leq |z| + |w|\,; & ||z| - |w|| \leq |z-w|\,;
\end{array}$$

- *and up to multiples of 2π then the following equations also hold:*
- $\arg(zw) = \arg z + \arg w$ if $z, w \neq 0$,
- $\arg(z/w) = \arg z - \arg w$ if $z, w \neq 0$,
- $\arg \bar{z} = -\arg z$ if $z \neq 0$.

A selection of the above statements is proved here; the remaining ones are left as exercises.

Proof. $|zw| = |z|\,|w|$. Let $z = a + bi$ and $w = c + di$. Then $zw = (ac - bd) + (bc + ad)\,i$ so that

$$\begin{aligned}
|zw| &= \sqrt{(ac-bd)^2 + (bc+ad)^2} \\
&= \sqrt{a^2c^2 + b^2d^2 + b^2c^2 + a^2d^2} \\
&= \sqrt{(a^2+b^2)(c^2+d^2)} \\
&= \sqrt{a^2+b^2}\sqrt{c^2+d^2} = |z|\,|w|\,.
\end{aligned}$$

■

Proof. $\arg(zw) = \arg z + \arg w$. Let $z = r(\cos\theta + i\sin\theta)$ and $w = R(\cos\Theta + i\sin\Theta)$. Then

$$\begin{aligned}
zw &= rR(\cos\theta + i\sin\theta)(\cos\Theta + i\sin\Theta) \\
&= rR((\cos\theta\cos\Theta - \sin\theta\sin\Theta) + i(\sin\theta\cos\Theta + \cos\theta\sin\Theta)) \\
&= rR(\cos(\theta+\Theta) + i\sin(\theta+\Theta))\,.
\end{aligned}$$

We can read off that $|zw| = rR = |z|\,|w|$ which is a second proof of the previous part and also that

$$\arg(zw) = \theta + \Theta = \arg z + \arg w \quad \text{up to multiples of } 2\pi.$$

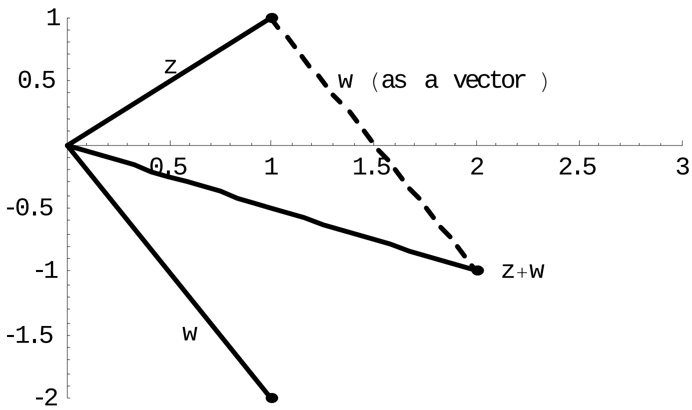
■

Proof. $\overline{zw} = \bar{z}\ \bar{w}$. Let $z = a + bi$ and $w = c + di$. Then

$$\begin{aligned}
\overline{zw} &= \overline{(ac-bd) + (bc+ad)\,i} \\
&= (ac-bd) - (bc+ad)\,i \\
&= (a-bi)(c-di) = \bar{z}\ \bar{w}.
\end{aligned}$$

■

Proof. (Triangle Inequality) $|z+w| \leq |z| + |w|$.



A diagrammatic proof of the Triangle Inequality

Note that the shortest distance between 0 and $z+w$ is the modulus of $z+w$. This is shorter in length than the path which goes from 0 to z to $z+w$. The total length of this second path is $|z| + |w|$.

For an algebraic proof, note that for any complex number $z + \bar{z} = 2\operatorname{Re} z$ and $\operatorname{Re} z \leq |z|$. So for $z, w \in \mathbb{C}$,

$$\frac{z\bar{w} + \bar{z}w}{2} = \operatorname{Re}(z\bar{w}) \leq |z\bar{w}| = |z|\,|\bar{w}| = |z|\,|w|.$$

Then

$$\begin{aligned} |z+w|^2 &= (z+w)\,\overline{(z+w)} \\ &= (z+w)\,(\bar{z}+\bar{w}) \\ &= z\bar{z} + z\bar{w} + \bar{z}w + w\bar{w} \\ &\leq |z|^2 + 2\,|z|\,|w| + |w|^2 = (|z|+|w|)^2, \end{aligned}$$

to give the required result. ■

Corollary 11 *The complex roots of a real polynomial come in pairs. That is, if z_0 satisfies the polynomial equation $a_k z^k + a_{k-1} z^{k-1} + \cdots + a_0 = 0$, where each a_i is real, then $\overline{z_0}$ is also a root.*

Proof. Note from the algebraic properties of the conjugate function, proven in the previous proposition, that

$$\begin{aligned} a_k\,(\overline{z_0})^k + a_{k-1}\,(\overline{z_0})^{k-1} + \cdots + a_1\overline{z_0} + a_0 &= a_k\overline{(z_0)^k} + a_{k-1}\overline{(z_0)^{k-1}} + \cdots + a_1\overline{z_0} + a_0 \\ &= \overline{a_k}\,\overline{(z_0)^k} + \overline{a_{k-1}}\,\overline{(z_0)^{k-1}} + \cdots + \overline{a_1}\,\overline{z_0} + \overline{a_0} \quad \text{[the } a_i \text{ are real]} \\ &= \overline{a_k\,(z_0)^k + a_{k-1}\,(z_0)^{k-1} + \cdots + a_0} \\ &= \overline{0} \quad [\text{as } z_0 \text{ is a root}] \\ &= 0. \end{aligned}$$

■

Exercise 17 ***A.*** *Let z and w be two complex numbers such that $zw = 0$. Show either $z = 0$ or $w = 0$.*

Exercise 18 ***A.*** *Suppose that the complex number α is a square root of z, that is $\alpha^2 = z$. Show that the only other square root of z is $-\alpha$. Suppose now that the complex numbers z_1 and z_2 have square roots $\pm\alpha_1$ and $\pm\alpha_2$ respectively. Show that the square roots of $z_1 z_2$ are $\pm\alpha_1\alpha_2$*

Exercise 19 ***B.*** *Prove the remaining identities from Proposition 10.*

Exercise 20 ***B.*** *Let t be a real number. Find expressions for*

$$x = \operatorname{Re} \frac{1}{2+ti}, \quad y = \operatorname{Im} \frac{1}{2+ti}.$$

Find an equation relating x and y by eliminating t. Deduce that the image of the line $\operatorname{Re} z = 2$ under the map $z \mapsto 1/z$ is contained in a circle. Is the image of the line all of the circle?

Exercise 21 ***B.*** *Let z_1 and z_2 be two complex numbers. Show that*

$$|z_1 - z_2|^2 + |z_1 + z_2|^2 = 2\left(|z_1|^2 + |z_2|^2\right).$$

This fact is called the Parallelogram Law — how does this relate the lengths of the diagonals and sides of the parallelogram? [Hint: consider the parallelogram in $\mathbb{C}$ with vertices $0, z_1, z_2, z_1 + z_2$.]

Exercise 22 ***C.*** *Consider a quadrilateral $OABC$ in the complex plane whose vertices are at the complex numbers $0, a, b, c$. Show that the equation*

$$|b|^2 + |a-c|^2 = |a|^2 + |c|^2 + |a-b|^2 + |b-c|^2$$

can be rearranged as

$$|b-a-c|^2 = 0.$$

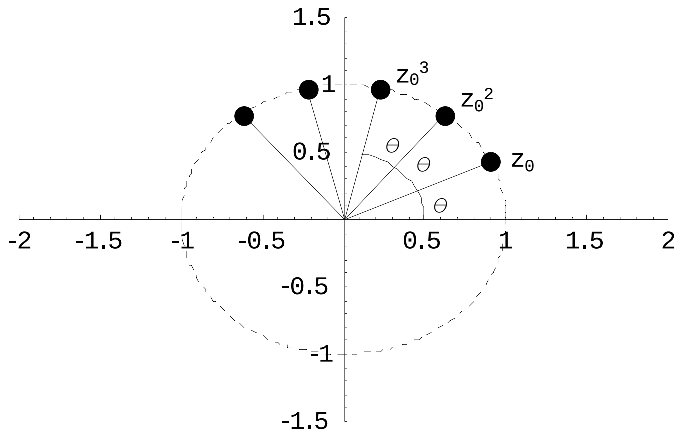
Hence show that the only quadrilaterals to satisfy the Parallelogram Law are parallelograms.

4 Roots Of Unity

Consider the complex number

$$z_0 = \cos\theta + i\sin\theta,$$

where $0 \leq \theta < 2\pi$. The modulus of z_0 is 1, and the argument of z_0 is θ.



Powers of z_0

In Proposition 10 we proved for $z, w \neq 0$ that

$$|zw| = |z|\,|w| \quad \text{and} \quad \arg(zw) = \arg z + \arg w,$$

up to multiples of 2π. So for any integer n, and any $z \neq 0$, we have that

$$|z^n| = |z|^n \quad \text{and} \quad \arg(z^n) = n \arg z.$$

So the modulus of $(z_0)^n$ is 1 and the argument of $(z_0)^n$ is $n\theta$, or putting this another way

Theorem 12 *(DE MOIVRE'S[5] THEOREM) For a real number θ and integer n we have that*

$$\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n .$$

Exercise 23 ***B.*** *Use De Moivre's Theorem to show that*

$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta,$$

and that

$$\sin 5\theta = \left(16\cos^4\theta - 12\cos^2\theta + 1\right)\sin\theta.$$

Exercise 24 ***B.*** *Let $z = \cos\theta + i\sin\theta$ and let n be an integer. Show that*

$$2\cos\theta = z + \frac{1}{z} \quad \text{and that} \quad 2i\sin\theta = z - \frac{1}{z}.$$

Find expressions for $\cos n\theta$ and $\sin n\theta$ in terms of z.

Exercise 25 ***B.*** *Show that*

$$\cos^5\theta = \frac{1}{16}\left(\cos 5\theta + 5\cos 3\theta + 10\cos\theta\right)$$

and hence find $\int_0^{\pi/2}\cos^5\theta\ d\theta$.

We apply these ideas to the following problem.

Problem 13 *Let n be a natural number. Find all those complex z such that $z^n = 1$.*

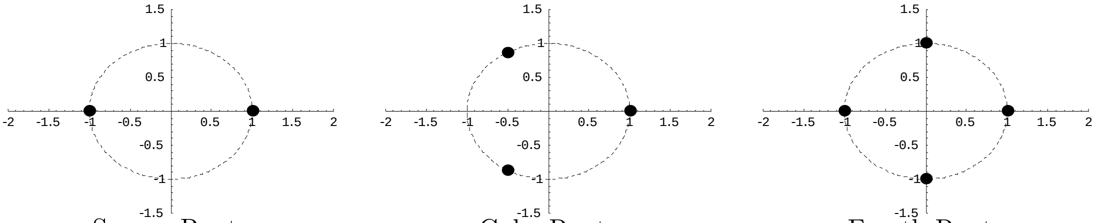
We know from the Fundamental Theorem of Algebra that there are (counting repetitions) n solutions: these are known as the nth roots of unity.

Let's first solve $z^n = 1$ directly for $n = 2, 3, 4$.

- When $n = 2$ we have
$$0 = z^2 - 1 = (z-1)(z+1)$$
and so the square roots of 1 are ± 1.
- When $n = 3$ we can factorise as follows
$$0 = z^3 - 1 = (z-1)\left(z^2 + z + 1\right).$$
So 1 is a root and completing the square we see
$$0 = z^2 + z + 1 = \left(z + \frac{1}{2}\right)^2 + \frac{3}{4}$$
which has roots
$$-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i.$$
So the cube roots of 1 are 1, $-1/2 + \sqrt{3}i/2$, and $-1/2 - \sqrt{3}i/2$.
- When $n = 4$ we can factorise as follows
$$0 = z^4 - 1 = \left(z^2 - 1\right)\left(z^2 + 1\right) = (z-1)(z+1)(z-i)(z+i),$$
so that the fourth roots of 1 are $1, -1, i$ and $-i$.

[5] De Moivre (1667-1754), a French protestant who moved to England, is best remembered for this formula, but his major contributions were in probability and appeared in his *The Doctrine Of Chances* (1718).

Plotting these roots on Argand diagrams we can see a pattern developing



Returning to the general case, suppose that $z = r(\cos\theta + i\sin\theta)$ and satisfies $z^n = 1$. Then by the observations preceding De Moivre's Theorem z^n has modulus r^n and has argument $n\theta$. As 1 has modulus 1 and argument 0, we can compare their moduli to find $r^n = 1$ giving $r = 1$. Comparing arguments, we see $n\theta = 0$ up to multiples of 2π. That is $n\theta = 2k\pi$ for some integer k, giving $\theta = 2k\pi/n$. So the roots of $z^n = 1$ are

$$z = \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right) \quad \text{where } k \text{ is an integer.}$$

At first glance there seem to be an infinite number of roots but note as cos and sin have period 2π then these z repeat with period n. So the nth roots of unity are

$$z = \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right) \quad \text{where} \quad k = 0, 1, 2, \ldots, n-1.$$

Plotted on an Argand diagram, the nth roots of unity form a regular n-gon inscribed within the unit circle with a vertex at 1.

Problem 14 *Find all the solutions of the cubic* $z^3 = -2 + 2i$.

If we write $-2 + 2i$ in its polar form we have

$$-2 + 2i = \sqrt{8}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right).$$

So if $z^3 = -2 + 2i$, and z has modulus r and argument θ, then

$$r^3 = \sqrt{8} \quad \text{and} \quad 3\theta = \frac{3\pi}{4} \quad \text{up to multiples of } 2\pi,$$

which gives

$$r = \sqrt{2} \quad \text{and} \quad \theta = \frac{\pi}{4} + \frac{2k\pi}{3} \quad \text{for some integer } k.$$

As before, we need only consider $k = 0, 1, 2$ (as other k lead to repeats) and we see the three roots are

$$\begin{aligned}
\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right) &= 1 + i, \\
\sqrt{2}\left(\cos\left(\frac{11\pi}{12}\right) + i\sin\left(\frac{11\pi}{12}\right)\right) &= \left(\frac{-1}{2} - \frac{\sqrt{3}}{2}\right) + i\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right), \\
\sqrt{2}\left(\cos\left(\frac{19\pi}{12}\right) + i\sin\left(\frac{19\pi}{12}\right)\right) &= \left(\frac{-1}{2} + \frac{\sqrt{3}}{2}\right) + i\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}\right).
\end{aligned}$$

Exercise 26 ***A.*** *Let ω be a cube root of unity (i.e. $\omega^3 = 1$) such that $\omega \neq 1$. Show that*

$$1 + \omega + \omega^2 = 0.$$

Exercise 27 ***C.*** *Let*

$$\zeta = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}.$$

Show that $\zeta^5 = 1$, and deduce that

$$1 + \zeta + \zeta^2 + \zeta^3 + \zeta^4 = 0.$$

Find the quadratic equation with roots $\zeta + \zeta^4$ and $\zeta^2 + \zeta^3$. Hence show that

$$\cos\frac{2\pi}{5} = \frac{\sqrt{5} - 1}{4}.$$

Exercise 28 ***C.*** *Determine the modulus and argument of the two complex numbers $1 + i$ and $\sqrt{3} + i$. Also write the number*

$$\frac{1 + i}{\sqrt{3} + i}$$

in the form $x + iy$. Deduce that

$$\cos\frac{\pi}{12} = \frac{\sqrt{3} + 1}{2\sqrt{2}} \quad \text{and} \quad \sin\frac{\pi}{12} = \frac{\sqrt{3} - 1}{2\sqrt{2}}.$$

Exercise 29 ***B.*** *Let $A = 1 + i$ and $B = 1 - i$. Find the two numbers C and D such that ABC and ABD are equilateral triangles in the Argand diagram. Show that if $C < D$ then*

$$A + \omega C + \omega^2 B = 0 = A + \omega B + \omega^2 D,$$

where $\omega = \left(-1 + \sqrt{3}i\right)/2$ is a cube root of unity other than 1.

Exercise 30 ***B.*** *By considering the seventh roots of -1 show that*

$$\cos\frac{\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{5\pi}{7} = \frac{1}{2}.$$

What is the value of

$$\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7}\,?$$

Exercise 31 ***B.*** *Find all the roots of the equation $x^8 = -1$. Hence, write $x^8 + 1$ as the product of four quadratic factors.*

Exercise 32 ***C.*** *Find all the roots of the following equations.*

1. $1 + z^2 + z^4 + z^6 = 0$,
2. $1 + z^3 + z^6 = 0$,
3. $(1 + z)^5 - z^5 = 0$,
4. $(z + 1)^9 + (z - 1)^9 = 0$.

5 Further Exercises

Exercise 33 ***C.*** *Given a non-zero complex number $z = x + iy$, we can associate with z a matrix*

$$Z = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}.$$

Show that if z and w are complex numbers with associated matrices Z and W, then the matrices associated with $z + w, zw$ and $1/z$ are $Z + W, ZW$ and Z^{-1} respectively. Hence, for each of the following matrix equations, find a matrix Z which is a solution.

$$\begin{aligned} Z^2 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\ Z^2 + 2Z &= \begin{pmatrix} -5 & 0 \\ 0 & -5 \end{pmatrix}, \\ Z^2 + \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix} Z &= \begin{pmatrix} -2 & -1 \\ 1 & -2 \end{pmatrix}, \\ Z^5 &= \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}. \end{aligned}$$

Exercise 34 ***C.*** *The sequence $x_0, x_1, x_2, x_3, \ldots$ is defined recursively by*

$$x_0 = 0, \quad x_1 = 0.8, \quad x_n = 1.2x_{n-1} - x_{n-2} \quad \text{for} \quad n \geq 2.$$

With the aid of a calculator list the values of x_i for $0 \leq i \leq 10$. Prove further, by induction, that

$$x_n = \operatorname{Im}\{(0.6 + 0.8i)^n\}$$

for each $n = 0, 1, 2, \ldots$. Deduce that $|x_n| \leq 1$ for all n. Show also that x_n cannot have the same sign for more than three consecutive n.

Exercise 35 ***C.*** *Consider the cubic equation $z^3 + mz + n = 0$ where m and n are real numbers. Let Δ be a square root of $(n/2)^2 + (m/3)^3$. We then define t and u by*

$$t = -n/2 + \Delta \quad \text{and} \quad u = n/2 + \Delta,$$

and let T and U respectively be cube roots of t and u. Show that tu is real, and that if T and U are chosen appropriately, then $z = T - U$ is a solution of the original cubic equation.

Use this method to completely solve the equation $z^3 + 6z = 20$. By making a substitution of the form $w = z - a$ for a suitable choice of a, find all three roots of the equation $8w^3 + 12w^2 + 54w = 135$.

Exercise 36 ***C.*** *Express $\tan 7\theta$ in terms of $\tan\theta$ and its powers. Hence solve the equation*

$$x^6 - 21x^4 + 35x^2 - 7 = 0.$$

Exercise 37 ***C.*** *Show for any complex number z, and any positive integer n, that*

$$z^{2n} - 1 = \left(z^2 - 1\right) \prod_{k=1}^{n-1} \left\{ z^2 - 2z\cos\frac{k\pi}{n} + 1 \right\}.$$

By setting $z = \cos\theta + i\sin\theta$ show that

$$\frac{\sin n\theta}{\sin\theta} = 2^{n-1} \prod_{k=1}^{n-1} \left\{ \cos\theta - \cos\frac{k\pi}{n} \right\}.$$